1. $\dfrac{\partial w}{\partial x} + f\left(\dfrac{\partial w}{\partial y}\right) = 0.$

This equation is encountered in optimal control and differential games.

1°. Complete integral:

$$w = C_1 y - f(C_1)x + C_2,$$

where C_1 and C_2 are arbitrary constants.

2°. On differentiating the equation with respect to y, we arrive at a quasilinear equation of the form 2.2.3:

$$\frac{\partial u}{\partial x} + f'(u)\frac{\partial u}{\partial y} = 0, \qquad u = \frac{\partial w}{\partial y}.$$

3°. The solution of the Cauchy problem with the initial condition $w(0, y) = \varphi(y)$ can be written in parametric form as

$$y = f'(\zeta)x + \xi, \quad w = \bigl[\zeta f'(\zeta) - f(\zeta)\bigr]x + \varphi(\xi), \qquad \text{where} \quad \zeta = \varphi'(\xi).$$

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