### 3. First-Order Nonlinear Partial Differential Equations

**Preliminary remarks.** For first-order partial differential equations with two independent variables, an exact solution

\[ w = \Phi(x, y, C_1, C_2) \]  

that depends on two arbitrary constants \( C_1 \) and \( C_2 \) is called a complete integral. The general integral (general solution) can be represented in parametric form by using the complete integral (1) and the two equations

\[
\begin{align*}
C_2 &= f(C_1), \\
\frac{\partial \Phi}{\partial C_1} + \frac{\partial \Phi}{\partial C_2} f'(C_1) &= 0,
\end{align*}
\]

where \( f \) is an arbitrary function and the prime stands for the derivative.

**References**


### 3.1. Equations Quadratic in One Derivative

1. \[ \frac{\partial w}{\partial x} + a \left( \frac{\partial w}{\partial y} \right)^2 = by. \]
2. \[ \frac{\partial w}{\partial x} + a \left( \frac{\partial w}{\partial y} \right)^2 + by^2 = 0. \]
3. \[ \frac{\partial w}{\partial x} + a \left( \frac{\partial w}{\partial y} \right)^2 = f(x) + g(y). \]
4. \[ \frac{\partial w}{\partial x} + a \left( \frac{\partial w}{\partial y} \right)^2 = f(x)y + g(x). \]
5. \[ \frac{\partial w}{\partial x} + a \left( \frac{\partial w}{\partial y} \right)^2 = f(x)w + g(x). \]
6. \[ \frac{\partial w}{\partial x} - f(w) \left( \frac{\partial w}{\partial y} \right)^2 = 0. \]
7. \[ f_1(x) \frac{\partial w}{\partial x} + f_2(y) \left( \frac{\partial w}{\partial y} \right)^2 = g_1(x) + g_2(y). \]
8. \[ \frac{\partial w}{\partial x} + a \left( \frac{\partial w}{\partial y} \right)^2 + b \frac{\partial w}{\partial y} = f(x) + g(y). \]
9. \[ \frac{\partial w}{\partial x} + a \left( \frac{\partial w}{\partial y} \right)^2 + b \frac{\partial w}{\partial y} = f(x)y + g(x). \]
10. \[ \frac{\partial w}{\partial x} + a \left( \frac{\partial w}{\partial y} \right)^2 + b \frac{\partial w}{\partial y} = f(x)w + g(x). \]
3.2. Equations Quadratic in Two Derivatives

1. \[ a \left( \frac{\partial w}{\partial x} \right)^2 + b \left( \frac{\partial w}{\partial y} \right)^2 = c. \]

2. \[ \left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 = a - 2by. \]

3. \[ \left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 = \frac{a}{\sqrt{x^2 + y^2}} + b. \]

4. \[ \left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 = f(x). \]

5. \[ \left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 = f(x) + g(y). \]

6. \[ \left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 = f(x^2 + y^2). \]

7. \[ \left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 = f(w). \]

8. \[ \left( \frac{\partial w}{\partial x} \right)^2 + \frac{1}{x^2} \left( \frac{\partial w}{\partial y} \right)^2 = f(x). \]

9. \[ \left( \frac{\partial w}{\partial x} \right)^2 + f(x) \left( \frac{\partial w}{\partial y} \right)^2 = g(x). \]

10. \[ \left( \frac{\partial w}{\partial x} \right)^2 + f(y) \left( \frac{\partial w}{\partial y} \right)^2 = g(y). \]

11. \[ \left( \frac{\partial w}{\partial x} \right)^2 + f(w) \left( \frac{\partial w}{\partial y} \right)^2 = g(w). \]

12. \[ f_1(x) \left( \frac{\partial w}{\partial x} \right)^2 + f_2(y) \left( \frac{\partial w}{\partial y} \right)^2 = g_1(x) + g_2(y). \]

3.3. Equations with Arbitrary Nonlinearities in Derivatives

1. \[ \frac{\partial w}{\partial x} + f \left( \frac{\partial w}{\partial y} \right) = 0. \]

2. \[ \frac{\partial w}{\partial x} + f \left( \frac{\partial w}{\partial y} \right) = g(x). \]

3. \[ \frac{\partial w}{\partial x} + f \left( \frac{\partial w}{\partial y} \right) = g(x)y + h(x). \]
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4. \( \frac{\partial w}{\partial x} + f\left( \frac{\partial w}{\partial y} \right) = g(x)w + h(x). \)

5. \( \frac{\partial w}{\partial x} - F\left( x, \frac{\partial w}{\partial y} \right) = 0. \)

6. \( \frac{\partial w}{\partial x} + F\left( x, \frac{\partial w}{\partial y} \right) = \alpha w. \)

7. \( \frac{\partial w}{\partial x} + F\left( x, \frac{\partial w}{\partial y} \right) = g(x)w. \)

8. \( F\left( \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y} \right) = 0. \)

9. \( w = x \frac{\partial w}{\partial x} + y \frac{\partial w}{\partial y} + F\left( \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y} \right). \)

10. \( F_1\left( x, \frac{\partial w}{\partial x} \right) = F_2\left( y, \frac{\partial w}{\partial y} \right). \)

11. \( F_1\left( x, \frac{\partial w}{\partial x} \right) + F_2\left( y, \frac{\partial w}{\partial y} \right) + \alpha w = 0. \)

12. \( F_1\left( x, \frac{1}{w} \frac{\partial w}{\partial x} \right) + w^kF_2\left( y, \frac{1}{w} \frac{\partial w}{\partial y} \right) = 0. \)

13. \( F_1\left( x, \frac{\partial w}{\partial x} \right) + e^{\lambda w}F_2\left( y, \frac{\partial w}{\partial y} \right) = 0. \)

14. \( F_1\left( x, \frac{1}{w} \frac{\partial w}{\partial x} \right) + F_2\left( y, \frac{1}{w} \frac{\partial w}{\partial y} \right) = k \ln w. \)

15. \( \frac{\partial w}{\partial x} + y F_1\left( x, \frac{\partial w}{\partial y} \right) + F_2\left( x, \frac{\partial w}{\partial y} \right) = 0. \)

16. \( F\left( \frac{\partial w}{\partial x} + \alpha y, \frac{\partial w}{\partial y} + \alpha x \right) = 0. \)

17. \( \left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 = F\left( x^2 + y^2, y \frac{\partial w}{\partial x} - x \frac{\partial w}{\partial y} \right). \)

18. \( F\left( x, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y} \right) = 0. \)

19. \( F\left( ax + by, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y} \right) = 0. \)

20. \( F\left( w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y} \right) = 0. \)

21. \( F\left( ax + by + cw, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y} \right) = 0. \)
22. \( F\left(x, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, w - y \frac{\partial w}{\partial y}\right) = 0. \)

23. \( F\left(w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, x \frac{\partial w}{\partial x} + y \frac{\partial w}{\partial y}\right) = 0. \)

24. \( F\left(ax + by, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, w - x \frac{\partial w}{\partial x} - y \frac{\partial w}{\partial y}\right) = 0. \)

25. \( F\left(x, \frac{\partial w}{\partial x}, G\left(y, \frac{\partial w}{\partial y}\right)\right) = 0. \)