3. First-Order Nonlinear Partial Differential Equations

Preliminary remarks. For first-order partial differential equations with two independent variables, an exact solution

$$w = \Phi(x, y, C_1, C_2) \tag{1}$$

that depends on two arbitrary constants C_1 and C_2 is called a complete integral. The general integral (general solution) can be represented in parametric form by using the complete integral (1) and the two equations

$$\begin{aligned} &C_2 = f(C_1), \\ &\frac{\partial \Phi}{\partial C_1} + \frac{\partial \Phi}{\partial C_2} f'(C_1) = 0, \end{aligned} \tag{2}$$

where f is an arbitrary function and the prime stands for the derivative.

References

Kamke, E., *Differentialgleichungen: Lösungsmethoden und Lösungen, II, Partielle Differentialgleichungen Erster Ordnung für eine gesuchte Funktion*, Akad. Verlagsgesellschaft Geest & Portig, Leipzig, 1965.

Polyanin, A. D., Zaitsev, V. F., and Moussiaux, A., *Handbook of First Order Partial Differential Equations* , Taylor & Francis, London, 2002.

3.1. Equations Quadratic in One Derivative

1. $\dfrac{\partial w}{\partial x} + a\left(\dfrac{\partial w}{\partial y}\right)^2 = by.$

2. $\dfrac{\partial w}{\partial x} + a\left(\dfrac{\partial w}{\partial y}\right)^2 + by^2 = 0.$

3. $\dfrac{\partial w}{\partial x} + a\left(\dfrac{\partial w}{\partial y}\right)^2 = f(x) + g(y).$

4. $\dfrac{\partial w}{\partial x} + a\left(\dfrac{\partial w}{\partial y}\right)^2 = f(x)y + g(x).$

5. $\dfrac{\partial w}{\partial x} + a\left(\dfrac{\partial w}{\partial y}\right)^2 = f(x)w + g(x).$

6. $\dfrac{\partial w}{\partial x} - f(w)\left(\dfrac{\partial w}{\partial y}\right)^2 = 0.$

7. $f_1(x)\dfrac{\partial w}{\partial x} + f_2(y)\left(\dfrac{\partial w}{\partial y}\right)^2 = g_1(x) + g_2(y).$

8. $\dfrac{\partial w}{\partial x} + a\left(\dfrac{\partial w}{\partial y}\right)^2 + b\dfrac{\partial w}{\partial y} = f(x) + g(y).$

9. $\dfrac{\partial w}{\partial x} + a\left(\dfrac{\partial w}{\partial y}\right)^2 + b\dfrac{\partial w}{\partial y} = f(x)y + g(x).$

10. $\dfrac{\partial w}{\partial x} + a\left(\dfrac{\partial w}{\partial y}\right)^2 + b\dfrac{\partial w}{\partial y} = f(x)w + g(x).$

3.2. Equations Quadratic in Two Derivatives

1. $a\left(\frac{\partial w}{\partial x}\right)^2 + b\left(\frac{\partial w}{\partial y}\right)^2 = c.$

2. $\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = a - 2by.$

3. $\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = \frac{a}{\sqrt{x^2+y^2}} + b.$

4. $\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = f(x).$

5. $\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = f(x) + g(y).$

6. $\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = f(x^2+y^2).$

7. $\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = f(w).$

8. $\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{x^2}\left(\frac{\partial w}{\partial y}\right)^2 = f(x).$

9. $\left(\frac{\partial w}{\partial x}\right)^2 + f(x)\left(\frac{\partial w}{\partial y}\right)^2 = g(x).$

10. $\left(\frac{\partial w}{\partial x}\right)^2 + f(y)\left(\frac{\partial w}{\partial y}\right)^2 = g(y).$

11. $\left(\frac{\partial w}{\partial x}\right)^2 + f(w)\left(\frac{\partial w}{\partial y}\right)^2 = g(w).$

12. $f_1(x)\left(\frac{\partial w}{\partial x}\right)^2 + f_2(y)\left(\frac{\partial w}{\partial y}\right)^2 = g_1(x) + g_2(y).$

3.3. Equations with Arbitrary Nonlinearities in Derivatives

1. $\frac{\partial w}{\partial x} + f\left(\frac{\partial w}{\partial y}\right) = 0.$

2. $\frac{\partial w}{\partial x} + f\left(\frac{\partial w}{\partial y}\right) = g(x).$

3. $\frac{\partial w}{\partial x} + f\left(\frac{\partial w}{\partial y}\right) = g(x)y + h(x).$

4. $\dfrac{\partial w}{\partial x} + f\left(\dfrac{\partial w}{\partial y}\right) = g(x)w + h(x).$

5. $\dfrac{\partial w}{\partial x} - F\left(x, \dfrac{\partial w}{\partial y}\right) = 0.$

6. $\dfrac{\partial w}{\partial x} + F\left(x, \dfrac{\partial w}{\partial y}\right) = aw.$

7. $\dfrac{\partial w}{\partial x} + F\left(x, \dfrac{\partial w}{\partial y}\right) = g(x)w.$

8. $F\left(\dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}\right) = 0.$

9. $w = x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} + F\left(\dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}\right).$

10. $F_1\left(x, \dfrac{\partial w}{\partial x}\right) = F_2\left(y, \dfrac{\partial w}{\partial y}\right).$

11. $F_1\left(x, \dfrac{\partial w}{\partial x}\right) + F_2\left(y, \dfrac{\partial w}{\partial y}\right) + aw = 0.$

12. $F_1\left(x, \dfrac{1}{w}\dfrac{\partial w}{\partial x}\right) + w^k F_2\left(y, \dfrac{1}{w}\dfrac{\partial w}{\partial y}\right) = 0.$

13. $F_1\left(x, \dfrac{\partial w}{\partial x}\right) + e^{\lambda w} F_2\left(y, \dfrac{\partial w}{\partial y}\right) = 0.$

14. $F_1\left(x, \dfrac{1}{w}\dfrac{\partial w}{\partial x}\right) + F_2\left(y, \dfrac{1}{w}\dfrac{\partial w}{\partial y}\right) = k \ln w.$

15. $\dfrac{\partial w}{\partial x} + yF_1\left(x, \dfrac{\partial w}{\partial y}\right) + F_2\left(x, \dfrac{\partial w}{\partial y}\right) = 0.$

16. $F\left(\dfrac{\partial w}{\partial x} + ay, \dfrac{\partial w}{\partial y} + ax\right) = 0.$

17. $\left(\dfrac{\partial w}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial y}\right)^2 = F\left(x^2 + y^2, y\dfrac{\partial w}{\partial x} - x\dfrac{\partial w}{\partial y}\right).$

18. $F\left(x, \dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}\right) = 0.$

19. $F\left(ax + by, \dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}\right) = 0.$

20. $F\left(w, \dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}\right) = 0.$

21. $F\left(ax + by + cw, \dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}\right) = 0.$

22. $F\left(x, \dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}, w - y\dfrac{\partial w}{\partial y}\right) = 0.$

23. $F\left(w, \dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}, x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y}\right) = 0.$

24. $F\left(ax + by, \dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}, w - x\dfrac{\partial w}{\partial x} - y\dfrac{\partial w}{\partial y}\right) = 0.$

25. $F\left(x, \dfrac{\partial w}{\partial x}, G\left(y, \dfrac{\partial w}{\partial y}\right)\right) = 0.$