21. $\displaystyle\int_0^x \ln(x-t)y(t)\,dt = f(x).$

Solution:

$$y(x) = -\int_0^x f''_{tt}(t)\,dt \int_0^\infty \frac{(x-t)^z e^{-\mathcal{C}z}}{\Gamma(z+1)}\,dz - f'_x(0)\int_0^\infty \frac{x^z e^{-\mathcal{C}z}}{\Gamma(z+1)}\,dz,$$

where $\mathcal{C} = \lim\limits_{k\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{k+1}-\ln k\right) = 0.5772\ldots$ is the Euler constant and $\Gamma(z)$ is the gamma function.

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