7. $\displaystyle y(x)-\lambda\int_0^x \frac{y(t)\,dt}{(x-t)^\alpha}=f(x), \qquad 0<\alpha<1.$

Generalized Abel integral equation of the second kind.

1°. Assume that the number α can be represented in the form

$$\alpha=1-\frac{m}{n}, \qquad \text{where} \quad m=1,2,\dots, \quad n=2,3,\dots \quad (m<n).$$

In this case, the solution of the generalized Abel equation of the second kind can be written in closed form (in quadratures):

$$y(x)=f(x)+\int_0^x R(x-t)f(t)\,dt,$$

where

$$R(x)=\sum_{\nu=1}^{n-1}\frac{\lambda^\nu\Gamma^\nu(m/n)}{\Gamma(\nu m/n)}x^{(\nu m/n)-1}+\frac{b}{m}\sum_{\mu=0}^{m-1}\varepsilon_\mu\exp\bigl(\varepsilon_\mu bx\bigr)$$
$$+\frac{b}{m}\sum_{\nu=1}^{n-1}\frac{\lambda^\nu\Gamma^\nu(m/n)}{\Gamma(\nu m/n)}\left[\sum_{\mu=0}^{m-1}\varepsilon_\mu\exp\bigl(\varepsilon_\mu bx\bigr)\int_0^x t^{(\nu m/n)-1}\exp\bigl(-\varepsilon_\mu bt\bigr)\,dt\right],$$
$$b=\lambda^{n/m}\Gamma^{n/m}(m/n), \qquad \varepsilon_\mu=\exp\left(\frac{2\pi\mu i}{m}\right), \qquad i^2=-1, \qquad \mu=0,1,\dots,m-1.$$

2°. Solution for any α from $0<\alpha<1$:

$$y(x)=f(x)+\int_0^x R(x-t)f(t)\,dt, \qquad \text{where} \quad R(x)=\sum_{n=1}^{\infty}\frac{\bigl[\lambda\Gamma(1-\alpha)x^{1-\alpha}\bigr]^n}{x\Gamma\bigl[n(1-\alpha)\bigr]}.$$

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