17. $\boldsymbol{y(x) + \int_a^x K(x-t)y(t)\,dt = f(x).}$

Renewal equation.

1°. To solve this integral equation, direct and inverse Laplace transforms are used. The solution can be represented in the form

$$y(x) = f(x) - \int_a^x R(x-t)f(t)\,dt. \qquad (1)$$

Here, the resolvent $R(x)$ is expressed via the kernel $K(x)$ of the original equation as follows:

$$R(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \widetilde{R}(p)e^{px}\,dp, \qquad \widetilde{R}(p) = \frac{\widetilde{K}(p)}{1+\widetilde{K}(p)}, \qquad \widetilde{K}(p) = \int_0^\infty K(x)e^{-px}\,dx.$$

2°. Let $w = w(x)$ be the solution of the simpler auxiliary equation with $a = 0$ and $f \equiv 1$:

$$w(x) + \int_0^x K(x-t)w(t)\,dt = 1. \qquad (2)$$

Then the solution of the original integral equation with arbitrary $f = f(x)$ is expressed via the solution of the auxiliary equation (2) as

$$y(x) = \frac{d}{dx}\int_a^x w(x-t)f(t)\,dt = f(a)w(x-a) + \int_a^x w(x-t)f_t'(t)\,dt.$$

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