22. $\displaystyle\int_{-\infty}^{\infty} K(x-t)y(t)\,dt = f(x).$

The Fourier transform is used to solve this equation.

1°. Solution:

$$y(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\widetilde{f}(u)}{\widetilde{K}(u)} e^{iux}\,du,$$

where

$$\widetilde{f}(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iux}\,dx, \qquad \widetilde{K}(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} K(x)e^{-iux}\,dx.$$

The following statement is valid. Let $f(x) \in L_2(-\infty,\infty)$ and $K(x) \in L_1(-\infty,\infty)$. Then for a solution $y(x) \in L_2(-\infty,\infty)$ of the integral equation to exist, it is necessary and sufficient that $\widetilde{f}(u)/\widetilde{K}(u) \in L_2(-\infty,\infty)$.

2°. Let the function $P(s)$ defined by the formula

$$\frac{1}{P(s)} = \int_{-\infty}^{\infty} e^{-st}K(t)\,dt$$

be a polynomial of degree n with real roots of the form

$$P(s) = \left(1-\frac{s}{a_1}\right)\left(1-\frac{s}{a_2}\right)\ldots\left(1-\frac{s}{a_n}\right).$$

Then the solution of the integral equation is given by

$$y(x) = P(D)f(x), \qquad D = \frac{d}{dx}.$$

References

Hirschman, I. I. and Widder, D. V., *The Convolution Transform*, Princeton Univ. Press, Princeton–New Jersey, 1955.
Ditkin, V. A. and Prudnikov, A. P., *Integral Transforms and Operational Calculus*, Pergamon Press, New York, 1965.
Polyanin, A. D. and Manzhirov, A. V., *Handbook of Integral Equations*, CRC Press, Boca Raton, 1998.

