Exact Solutions of the Generalized Equal Width Wave Equation

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Abstract. The equal width wave (EW) equation is a model partial differential equation for the simulation of one-dimensional wave propagation in nonlinear media with dispersion processes. The EW-Burgers equation models the propagation of nonlinear and dispersive waves with certain dissipative effects. In this work, we derive exact solitary wave solutions for the general form of the EW equation and the generalized EW-Burgers equation with nonlinear terms of any order. We also derive analytical expressions of three invariants of motion for solitary wave solutions of the generalized EW equation.

1 Introduction

The well-known Korteweg and de Vries (KdV) equation, $u_t + uu_x + u_{xxx} = 0$, is a nonlinear partial differential equation (PDE) that models the time-dependent motion of shallow water waves in one space dimension. Morrison et al. [4] proposed the one-dimensional PDE, $u_t + uu_x - \mu u_{xxt} = 0$, as an equally valid and accurate model for the same wave phenomena simulated by the KdV equation. This PDE is called the equal width (EW) equation because the solutions for solitary waves with a permanent form and speed, for a given value of the parameter μ, are waves with an equal width or wavelength for all wave amplitudes.

In this paper we present exact solitary wave solutions for the EW equation in its general form with nonlinear terms of any order

$$u_t + au^p u_x - \mu u_{xxt} = 0 \ , \tag{1}$$

and the generalized EW-Burgers equation with nonlinear terms of any order and dissipative effects modeled by the term δu_{xx},

$$u_t + au^p u_x - \delta u_{xx} - \mu u_{xxt} = 0 \ . \tag{2}$$

The solutions for (1) are obtained by integrating a first order nonlinear ODE using `Maple`. Solutions for (2) are difficult to derive by integrating directly the associated first order ODE. We used an approach devised by Zhang et al. [7] which is based on assuming a certain form of the solitary wave solution. The solutions are obtained by integrating a second order nonlinear ODE using a technique similar to that used by Kichennassamy and Olver [3] to transform a nonlinear ordinary differential equation to a homogeneous polynomial that can be solved using `Maple`. The approach presented in this work is general and can also be applied for finding exact solutions for other nonlinear wave equations such as Boussinesq-like equations (Hamdi et al. [2]). The exact solitary wave solutions can be used to specify initial data for the incident waves in the numerical model and for the verification of the associated computed solution [1]. We also derive analytical expressions for three invariants of motion which can be used as verification tools to investigate the conservation properties of the numerical method.

2 Derivation of the Exact Solutions

We concentrate on finding an exact solitary wave solution of the form

$$u(x,t) = u(x - x_0 - Ct) \ . \tag{3}$$

This corresponds to a traveling-wave propagating with steady celerity C. We are interested in solutions depending only on the moving coordinate $\xi = x - x_0 - Ct$ as,

$$u(x,t) = u(x - x_0 - Ct) \equiv u(\xi) \ . \tag{4}$$

Substituting into (2), the function $u(\xi)$ satisfies a third order nonlinear ordinary differential equation (ODE),

$$-Cu' + au^p u' - \delta u'' + \mu C u''' = 0 \ , \tag{5}$$

where the derivatives are performed with respect to the coordinate ξ.

Integrating once, we obtain

$$-Cu + \frac{a}{p+1} u^{p+1} - \delta u' + \mu C u'' = k_1 \ . \tag{6}$$

where k_1 is a constant of integration. If we assume that the solitary wave solution and its derivatives have the following asymptotic values.

$$u(\xi) \longrightarrow u_\pm \ \text{ as } \ \xi \longrightarrow \pm\infty \ , \tag{7}$$

and for $n \geq 1$

$$u^{(n)}(\xi) \longrightarrow 0 \ \text{ as } \ \xi \longrightarrow \pm\infty \ . \tag{8}$$

We also assume that $u_\pm$ satisfies the following algebraic equation

$$-Cu_\pm + \frac{a}{p+1} u_\pm^{p+1} = 0 \, , \tag{9}$$

then the integration constant k_1 is equal to zero and (6) reduces to

$$-Cu + \frac{a}{p+1} u^{p+1} - \delta u' + \mu C u'' = 0 \ . \tag{10}$$

From (9) we also have the relation

$$\frac{a}{p+1}\left(u_+^{p+2} - u_-^{p+2}\right) = C\left(u_+^2 - u_-^2\right) \ . \tag{11}$$

To allow another integration, we first multiply (10) by $2u'$. Then each term can be integrated separately to obtain,

$$-Cu^2 + \frac{2a}{(p+1)(p+2)} u^{p+2} - 2\delta \int_{-\infty}^{\xi} (u')^2 \, \mathrm{d}\xi + \mu C (u')^2 = k_2 \ , \tag{12}$$

where k_2 is a second constant of integration.

Using the asymptotic boundary conditions (7) and (8) at infinity we have

$$-Cu_+^2 + \frac{2a}{(p+1)(p+2)} u_+^{p+2} - 2\delta \int_{-\infty}^{+\infty} (u')^2 \, \mathrm{d}\xi = k_2, \text{ as } \xi \longrightarrow +\infty \ , \tag{13}$$

and

$$-Cu_-^2 + \frac{2a}{(p+1)(p+2)} u_-^{p+2} = k_2, \text{ as } \xi \longrightarrow -\infty \ . \tag{14}$$

By substituting (14) into (13) and using the relation (11), we obtain the analytical expression of the following important square integral,

$$\int_{-\infty}^{+\infty} (u')^2 \, \mathrm{d}\xi = \frac{C}{2\delta} \frac{p}{p+2} \left(u_-^2 - u_+^2\right) \ . \tag{15}$$

In constructing solitary wave solutions for (1) and (2) with the assumptions (7), (8) and (9) the following important results must be considered.

1. The integral $\int_{-\infty}^{+\infty} (u')^2 \, \mathrm{d}\xi$ is well defined and the function $u'(\xi)$ is square integrable over the domain $[-\infty, +\infty]$.
2. The generalized EW-Burgers equation (2) with dissipative effects ($\delta > 0$) does not have bell-profile solitary wave solutions. According to (15) solitary waves solutions exist if and only if $|u_-| > |u_+|$ because $\int_{-\infty}^{+\infty} (u')^2 \, \mathrm{d}\xi > 0$. The trivial case $|u_-| = |u_+|$ does not lead to solitary wave solutions because the solution is constant ($u'(\xi) = 0$) since $\int_{-\infty}^{+\infty} (u')^2 \, \mathrm{d}\xi = 0$. The EW-Burgers equation (2) with ($\delta > 0$) has no solutions if $|u_-| < |u_+|$.
3. The generalized EW equation (1) with no dissipation ($\delta = 0$) has bell-profile solitary waves which satisfy the condition $|u_-| = |u_+|$.

2.1 Solitary Wave Solutions of the Generalized EW Equation

Setting $\delta = 0$ (no dissipation) in (12) leads to the following nonlinear ODE,

$$-Cu^2 + \frac{2a}{(p+1)(p+2)} u^{p+2} + \mu C (u')^2 = k_2 \; . \tag{16}$$

If we also assume that $|u_-| = |u_+| = 0$ then the second integration constant k_2 is equal to zero. There are several analytical techniques for integrating this type of nonlinear ODEs [2]. These ODEs can also be solved using symbolic computation. First we make the following change of the dependent variable in (16)

$$u(\xi) = \phi^{1/p}(\xi) \tag{17}$$

to reduce the power of the nonlinear term u^{p+2} in (16) . Using the above transformation we obtain,

$$-C\phi^2 + \frac{2a}{(p+1)(p+2)} \phi^3 + \frac{\mu C}{p^2} (\phi')^2 = 0 \; . \tag{18}$$

Let $\alpha = C, \beta = \dfrac{2a}{(p+1)(p+2)}$ and $\gamma = \dfrac{\mu C}{p^2}$, then we have

$$-\alpha\phi^2 + \beta\phi^3 + \gamma(\phi')^2 = 0 \; , \tag{19}$$

which can be solved using the following `Maple` script

```
> ode:=-alpha*phi(xi)^2+beta*phi(xi)^3+gamma*diff(phi(xi),xi)^2;
> dsolve(ode1);
```

which returns the following exact solution

$$\phi(\xi) = \frac{\alpha}{\beta} \left[1 - \tanh^2 \left(\frac{1}{2} \sqrt{\frac{\alpha}{\gamma}} \, (\xi - x_0) \right) \right] = \frac{4\alpha}{\beta} \frac{e^{\sqrt{\frac{\alpha}{\gamma}}(\xi - x_0)}}{\left[e^{\sqrt{\frac{\alpha}{\gamma}}(\xi - x_0)} + 1 \right]^2} \; , \tag{20}$$

in which x_0 is a constant of integration. From trigonometry we have the identity $\operatorname{sech}^2 = 1 - \tanh^2$. It follows that

$$\phi(\xi) = \frac{\alpha}{\beta} \operatorname{sech}^2 \left(\frac{1}{2} \sqrt{\frac{\alpha}{\gamma}} \, (\xi - x_0) \right) . \tag{21}$$

After substitutions ($\xi = x - C\,t$, $\alpha = C, \beta = 2a/(p+1)(p+2)$, $\gamma = \mu C/p^2$ and $u(\xi) = \phi^{1/p}(\xi)$), we obtain an explicit expression of the exact solution for the generalized EW equation:

$$u(x,t) = \left[\frac{(p+1)(p+2)}{2a} \, C \operatorname{sech}^2 \left(\frac{1}{2} p \sqrt{\frac{1}{\mu}} \, (x - C\,t - x_0) \right) \right]^{\dfrac{1}{p}} \; . \tag{22}$$

This analytical solution is a bell-profile solitary wave. It is a single pulse of amplitude $[C(p+1)(p+2)/2a]^{1/p}$ initially centered at x_0 and traveling without change of shape at a steady celerity C and a wave number $\frac{1}{2}p\sqrt{1/\mu}$. The wave number also characterizes the width of the solitary wave. Unlike solitary waves of the KdV, the wave number of the EW equation is independent of the wave amplitude and the wave celerity. For a given value of the parameter μ all solitary waves of the EW equation have an equal width for all wave amplitudes and wave celerities. This is the special feature for which the Equal Width equation was named.

2.2 Solitary Wave Solutions of the Generalized EW-Burgers Equation

The dissipation effects in the generalized EW-Burgers equation are modeled by the term δu_{xx}. Solutions based on the integration of the first order ODE (12) with $\delta > 0$ are difficult to obtain because the square integral $\int_{-\infty}^{\xi}(u')^2\,\mathrm{d}\xi$ cannot be expressed in explicit form. The solutions will be obtained using the second order ODE (10) instead, and by assuming a certain form for the solitary wave to transform this ODE to a third order polynomial. First we choose a particular change for the dependent variable $u(\xi) = \phi^{2/p}(\xi)$ to reduce the power of the nonlinear term u^{p+1} in (10). The change of variable $u(\xi) = \phi^{1/p}(\xi)$ used previously for the derivation of exact solutions for the EW equation will lead to a nonlinear ODE that cannot be solved analytically. After transformations and simplifications we obtain

$$-C\phi^2 + \frac{a}{(p+1)}\phi^4 - \frac{2\delta}{p}\phi\phi' + \frac{2\mu\, C}{p}\phi\phi'' + \frac{2\mu\, C(2-p)}{p^2}(\phi')^2 = 0\,. \tag{23}$$

The integration of the above ODE is not straightforward because all terms are nonlinear. In such situations, the following integration technique can be used.

Since we are seeking kink-profile solitary wave solutions that satisfy the condition $|u_-| > |u_+|$ and the assumptions (7), (8) and (9), we will assume that such solutions have the form,

$$\phi(\xi) = \frac{\phi_0}{2}\,\left[1 - \tanh\left(\,\kappa\,(\xi - x_0)\right)\right] = \frac{\phi_0}{e^{2\,\kappa\,(\xi - x_0)} + 1}\quad . \tag{24}$$

Using some recursive properties of the function `tanh`, we can express the derivative terms ϕ', ϕ'' and $(\phi')^2$ which appear in ODE (23) as homogeneous polynomials of the dependent variable ϕ only.

$$\phi' = -2\kappa\phi + 2\frac{\kappa}{\phi_0}\phi^2\,, \tag{25}$$

$$\phi'' = 4\kappa^2\phi - 12\frac{\kappa^2}{\phi_0}\phi^2 - 8\frac{\kappa^2}{\phi_0^2}\phi^3\,, \tag{26}$$

$$(\phi')^2 = 4\kappa^2\phi^2 - 8\frac{\kappa^2}{\phi_0}\phi^3 + 4\frac{\kappa^2}{\phi_0^2}\phi^4\ . \tag{27}$$

Now substituting (25), (26) and (27) into (23) leads to a quartic homogeneous polynomial in u on its left-hand side,

$$\begin{aligned}&\left(-C + \frac{4\,\delta\,\kappa}{p} + \frac{8\,\mu\,C\,(2-p)\,\kappa^2}{p^2} + \frac{8\,\mu\,C\,\kappa^2}{p}\right)\phi^2\\ &\quad + \left(-24\,\frac{\mu\,C\,\kappa^2}{p\,\phi_0} - \frac{4\,\delta\,\kappa}{p\,\phi_0} - \frac{16\,\mu\,C\,(2-p)\,\kappa^2}{p^2\,\phi_0}\right)\phi^3\\ &\qquad + \left(\frac{a}{p+1} + \frac{16\,\mu\,C\,\kappa^2}{p\,\phi_0^2} + \frac{8\,\mu\,C\,(2-p)\,\kappa^2}{p^2\,\phi_0^2}\right)\phi^4\ .\end{aligned} \tag{28}$$

It follows from equation (28) that all the coefficients of the polynomial must be zero in order to obtain a nontrivial solution ϕ. Setting the coefficients to zero yields a nonlinear algebraic system of three equations for the three unknowns C, κ and ϕ_0,

$$\begin{cases} -C + \dfrac{4\,\delta\,\kappa}{p} + \dfrac{8\,\mu\,C\,(2-p)\,\kappa^2}{p^2} + \dfrac{8\,\mu\,C\,\kappa^2}{p} = 0\,,\\[2ex] -24\,\dfrac{\mu\,C\,\kappa^2}{p\,\phi_0} - \dfrac{4\,\delta\,\kappa}{p\,\phi_0} - \dfrac{16\,\mu\,C\,(2-p)\,\kappa^2}{p^2\,\phi_0} = 0\,,\\[2ex] \dfrac{a}{p+1} + \dfrac{16\,\mu\,C\,\kappa^2}{p\,\phi_0^2} + \dfrac{8\,\mu\,C\,(2-p)\,\kappa^2}{p^2\,\phi_0^2} = 0\,. \end{cases} \tag{29}$$

The analytical solution of the system (29) can be obtained using `Maple`. If we consider the generalized EW-Burgers equation $u_t + au^p u_x \pm \delta u_{xx} \pm \mu u_{xxt} = 0$, with arbitrary signs for the dissipative and dispersive parameters then the solution of a more general form of the system (29) provides the following exact expressions for the celerity C, wave number κ and wave amplitude ϕ_0.

$$C = \pm\,\frac{1}{2\mu\,(p+4)}\,\frac{\delta}{\sqrt{\pm\dfrac{1}{8\,\mu p + 16\,\mu}}}\ , \tag{30}$$

$$\kappa = \pm\,p\,\sqrt{\pm\frac{1}{8\,\mu p + 16\,\mu}}\ , \tag{31}$$

$$\phi_0 = \pm 2\sqrt{\pm\frac{\delta\sqrt{\pm 2\,\dfrac{1}{\mu\,(p+2)}}\,p^2 + 3\,\delta\sqrt{\pm 2\,\dfrac{1}{\mu\,(p+2)}}\,p + 2\,\delta\sqrt{\pm 2\,\dfrac{1}{\mu\,(p+2)}}}{4\,a\,p + 16\,a}}\ . \tag{32}$$

From (24) and the relation ($u(\xi) = \phi^{2/p}(\xi)$) we finally deduce the explicit expressions for the exact solution

$$\phi(\xi) = \left\{\frac{\phi_0}{2}\,\left[1 - \tanh\left(\,\kappa\,(\xi - x_0)\right)\right]\right\}^{2/p} = \left[\frac{\phi_0}{e^{2\,\kappa\,(\xi - x_0)} + 1}\right]^{2/p}\ . \tag{33}$$

Using the identity $\mathrm{sech}^2 = 1 - \tanh^2$ the solution can also be written as

$$u(x,t) = \left\{ \phi_0 \left[1 - \tanh\left(\kappa\,(\xi - C\,t - x_0)\right) - \frac{1}{2}\mathrm{sech}^2\left(\kappa\,(\xi - C\,t - x_0)\right)\right]\right\}^{1/p} . \tag{34}$$

This monotonic function is a kink-profile solitary wave. Similar profiles modeling the propagation of undular bores were studies numerically by Hamdi et al. [1] using the EW equation. It is easy to verify that the solution (34) satisfies (2) and the assumptions (7), (8) and (9)

INVARIANTS OF MOTION OF SOLITARY WAVE SOLUTIONS

Olver [5] has shown that solutions of the EW equation ($u_t + uu_x - \mu u_{xxt} = 0$) have only three non-trivial conservation laws that can be written in the general form $T_t + X_x = 0$. These laws are the equivalents of the conservation of mass, momentum and energy in fluid mechanics. Olver showed that the three laws lead directly to three so-called invariants of motion given by

$$C_1 \;=\; \int_{-\infty}^{+\infty} u \,\mathrm{d}x, \quad C_2 \;=\; \int_{-\infty}^{+\infty} \left(u^2 + \mu u_x u_x \right) \mathrm{d}x, \quad C_3 \;=\; \int_{-\infty}^{+\infty} u^3 \,\mathrm{d}x, \tag{35}$$

provided that the integrals converge. These invariants of motion for the EW equation need to be extended for the general case $u_t + au^p u_x - \mu u_{xxt} = 0$ and also generalized for the case of a finite spatial domain with asymptotic boundary conditions (7) and (8). This is done by multiplying the generalized EW equation, $u_t + au^p u_x - \mu u_{xxt} = 0$, by 1, u and $\left(2\frac{a}{p+1}u^{p+1} - 2\mu u_{xt}\right)$, and then the resulting three equations can each be expressed in the form $T_t + X_x = 0$ as

$$(u)_t + \left(\frac{a}{(p+1)} u^{p+1} - \mu u_{xt} \right)_x = 0\,,$$

$$\left(u^2 + \mu u_x u_x\right)_t + \left(2\frac{a}{(p+2)} u^{p+2} - 2\mu u u_{xt} \right)_x = 0\,,$$

$$\left(\frac{2\,a\,u^{p+2}}{(p+1)(p+2)} \right)_t + \left(\frac{a^2\,u^{2(p+1)}}{(p+1)^2} - \mu u_t u_t - \frac{2\mu\,a\,u^{p+1} u_{xt}}{(p+1)} + \mu^2 u_{xt} u_{xt} \right)_x = 0\,.$$

These three conservation laws can now be integrated easily with respect to x over a large but finite spatial domain $[x_L, x_U]$ instead of $[-\infty, +\infty]$ to obtain the intermediate results

$$\frac{\partial}{\partial t} \int_{x_L}^{x_U} u \,\mathrm{d}x + \frac{a}{(p+1)} \left(u_U^{(p+1)} - u_L^{(p+1)} \right) \;=\; 0\,,$$

$$\frac{\partial}{\partial t} \int_{x_L}^{x_U} \left[u^2 + \mu u_x u_x \right] \mathrm{d}x + \frac{2\,a}{(p+2)} \left(u_U^{(p+2)} - u_L^{(p+2)} \right) \;=\; 0\,, \tag{39}$$

$$\frac{\partial}{\partial t}\int_{x_L}^{x_U}\left(\frac{2\,a}{(p+2)}\,u^{p+2}\right)\,\mathrm{d}x+\frac{a^2}{(p+1)}\,u^{2(p+1)}\left(u_U^{2(p+1)}-u_L^{2(p+1)}\right)\;=\;0\,,$$

after simplifications. In these equations, $u_L = u(x_L, t)$ and $u_U = u(x_U, t)$ are time-invariant constants at the domain boundaries. In the simplifications, the terms $[u_{xt}]_{x_L}^{x_U}$, $[uu_{xt}]_{x_L}^{x_U}$, $[u_t u_t]_{x_L}^{x_U}$, $[u^2 u_{xt}]_{x_L}^{x_U}$ and $[u_{xt}u_{xt}]_{x_L}^{x_U}$ are zero at the boundaries (assumptions (7) and (8)). Equation (39) can now be integrated with respect to t to yield

$$\begin{aligned}
C_1 \;&=\; \int_{x_L}^{x_U} u\,\mathrm{d}x+\frac{a}{(p+1)}\left(u_U^{(p+1)}-u_L^{(p+1)}\right)t\,,\\
C_2 \;&=\; \int_{x_L}^{x_U}\left[u^2+\mu u_x u_x\right]\mathrm{d}x+\frac{2\,a}{(p+2)}\left(u_U^{(p+2)}-u_L^{(p+2)}\right)t\,,\\
C_3 \;&=\; \int_{x_L}^{x_U}\left(\frac{2\,a}{(p+2)}\,u^{p+2}\right)\mathrm{d}x+\frac{a^2}{(p+1)}\,u^{2(p+1)}\left(u_U^{2(p+1)}-u_L^{2(p+1)}\right)t\,.
\end{aligned}\tag{40}$$

These invariants of motion are generalizations of those given by equation (35), extended for the case of a large but finite length spatial domain when the solution of $u(x, t)$ of the generalized EW equation is constant but not necessarily zero at the domain boundaries (assumptions (7) and (8)). The extra terms stem directly from the convection of mass, momentum and energy into and out of the lower and upper boundaries of the spatial domain. These invariants of motion are equal to the initial ($t = 0$) mass, momentum and energy inside the domain $[x_L, x_U]$. Note that during numerical computations that provide solutions to the EW equation, C_1, C_2 and C_3 can be calculated after each successive time step over the entire spatial domain $x_L \leq x \leq x_U$ that contains the wave motion, such that the conservation properties of the numerical algorithm can be monitored and thereby assessed.

In this section we will derive analytic expressions for the three invariants of motion corresponding to the solitary wave solutions of the generalized EW equation $u_t + au^p u_x - \mu u_{xxt} = 0$. For simplicity, the invariants are given for $p = 1$. However, the evaluation of these invariants for any given value of the order p ($p \geq 1$) is straightforward (simple change of the value of p in the `Maple` script). These invariants, also called constants of motion, are independent of time and hence have the same value at any time T for waves initially location at any position X_0. Therefore, these invariants will be evaluated at the initial state $T = 0$. We also suppose that the solitary wave is initially located at $X_0 = 0$ and $[x_L, x_U] = [-L, L]$.

The first invariant of motion corresponds to the conservation of mass,

$$I_1 = \int_{-\infty}^{+\infty} u\,\mathrm{d}x\ \ . \tag{41}$$

The conservation of mass over the spatial domain $[-L, L]$ is given by

$$I_1(L) = \int_{-L}^{+L} u \, \mathrm{d}x \quad . \tag{42}$$

Integrating (22) using `Maple` we have,

```
> u:=(((p+1)*(p+2)/(2*a))*C* \
     (sech((1/2)*p*sqrt(1/mu)*(x-X0-C*T)))^2)^(1/p):
> p:=1:
> X0:=0:
> T:=0:
> I_1:=int(u,x=-L..L):
```

$$I_1(L) = \int_{-L}^{L} u(x,t) \, \mathrm{d}x = 12 \, \frac{\sinh(\frac{1}{2}\sqrt{\frac{1}{\mu}}\, L)}{a\sqrt{\frac{1}{\mu}}\cosh(\frac{1}{2}\sqrt{\frac{1}{\mu}}\, L)} \quad . \tag{43}$$

The conservation of mass over the whole domain $[-\infty, +\infty]$ is given by,

```
> simplify(limit(I_1, L=infinity)):
```

$$\int_{-\infty}^{\infty} u(x,t) \, \mathrm{d}x = \lim_{L\to+\infty} I_1(L) = 12 \, \frac{\sqrt{C}\,\sqrt{\mu}}{a} \quad . \tag{44}$$

The second invariant of motion represents the conservation of energy,

$$I_2 = \int_{-\infty}^{+\infty} \left(u^2 + \mu u_x u_x\right) \, \mathrm{d}x \quad . \tag{45}$$

The conservation of energy over the spatial domain $[-L, L]$ is given by

$$I_2(L) = \int_{-L}^{L} \left(u^2 + \mu u_x u_x\right) \, \mathrm{d}x. \tag{46}$$

Using `Maple`, we obtain,

```
> I_2:=int(u^2 + mu*(u_x)^2,x=-L..L):
```

$$I_2(L) = \frac{36}{5} \, \frac{C^2 \sinh(\frac{1}{2}\sqrt{\frac{1}{\mu}}\, L)\,(2\cosh(\frac{1}{2}\sqrt{\frac{1}{\mu}}\, L)^2 + 4\cosh(\frac{1}{2}\sqrt{\frac{1}{\mu}}\, L)^4 - 1)}{a^2\sqrt{\frac{1}{\mu}}\cosh(\frac{1}{2}\sqrt{\frac{1}{\mu}}\, L)^5} \, , \tag{47}$$

The conservation of energy over the whole domain $[-\infty, +\infty]$ is given by

```
> simplify(limit(I_2, L=infinity)):
```

$$\int_{-\infty}^{+\infty} \left(u^2 + \mu u_x u_x\right)\, \mathrm{d}x \quad = \lim_{L\to+\infty} I_2(L) = \frac{144}{5} \frac{C^2\sqrt{\mu}}{a^2} \quad . \tag{48}$$

The third invariant, which is similar to the invariant discovered by Whitham [6], is given by

$$I_3 = \int_{-\infty}^{+\infty} u^3\, \mathrm{d}x \quad . \tag{49}$$

The third invariant considered over the spatial domain $[-L, L]$ reduces to

$$I_3(L) = \int_{-L}^{+L} u^3\, \mathrm{d}x \quad . \tag{50}$$

Using `Maple`, we obtain,

```
> I_3:=int(u^3,x=-L..L):
```

$$I_3(L) = \frac{36}{5} \frac{C^3 \sinh(\frac{1}{2}\sqrt{\frac{1}{\mu}}\,L)\,(3 + 4\cosh(\frac{1}{2}\sqrt{\frac{1}{\mu}}\,L)^2 + 8\cosh(\frac{1}{2}\sqrt{\frac{1}{\mu}}\,L)^4)}{a^3\sqrt{\frac{1}{\mu}}\cosh(\frac{1}{2}\sqrt{\frac{1}{\mu}}\,L)^5} \tag{51}$$

The third invariant evaluated over the whole domain $[-\infty, +\infty]$ is given by

```
> simplify(limit(I_3, L=infinity)):
```

$$\int_{-\infty}^{+\infty} u^3\, \mathrm{d}x = \lim_{L\to+\infty} I_3(L) = \frac{288}{5} \frac{C^3\sqrt{\mu}}{a^3} \quad .$$

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