Be careful with Exp-function method

Nikolai A. Kudryashov, Nadejda B. Loguinova

Department of Applied Mathematics, Moscow Engineering and Physics Institute (State university), 31 Kashirskoe Shosse, 115409, Moscow, Russian Federation

Department of Computational Mathematics, Moscow State University of Transport Communications, 15 Obraztsova Street, 127994 Moscow, Russian Federation

Abstract

An application of the Exp-function method to search for exact solutions of nonlinear differential equations is analyzed. Typical mistakes of application of the Exp-function method are demonstrated. We show it is often required to simplify the exact solutions obtained. Possibilities of the Exp-function method and other approaches in mathematical physics are discussed. The application of the singular manifold method for finding exact solutions of the Fitzhugh - Nagumo equation is illustrated. The modified simple equation method is introduced. This approach is used to look for exact solutions of the generalized Korteweg - de Vries equation.

Keywords: Nonlinear evolution equation, Exact solution, Exp-function method, Simplest equation method

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1 Introduction

He and Wu introduced the so called Exp-function method in [1] to search for exact solutions of nonlinear differential equations. At present this method is very popular and we can find a lot of publications with applications of this method in journals [2–15]. We can read a number of ecstatic words about possibilities of this method. Let us present here some of them.

"The expression of the Exp-function method is more general than the sinh-function and tanh-function, so we can find more general solutions in Exp-function method" [10].

"Exp-function method is easy, concise and an effective method to implement to nonlinear evolution equations arising in mathematical physics" [11].

"All applications verified that the Exp-function method is straightforward, concise and effective in obtaining generalized solitary solutions and periodic solutions of nonlinear evolution equations. The main merits of this method over the other methods are that it gives more general solutions with some free parameters" [12].

"The solution procedure of this method can be easily extended to other kinds of nonlinear evolution equations" [13].

"The Exp-function method leads to not only generalized solitonary solutions but also periodic solutions" [14].

"Our first interest is implementing the Exp-function method to stress its power in handling nonlinear equations, so that one can apply it to models of various types of nonlinearity" [15].

We hope we have enough raptures about the Exp-function method.

The aim of this paper is to analyze some recent papers with application of the Exp-function method and to discuss the main deficiencies of this method.

We would like also to point out the niche which this method can occupy in comparison with other methods for finding exact solutions of nonlinear differential equations.

The idea of the Exp-function method is very simple [2–15] but results are very cumbersome. If we look at papers [5–7] we will note that we can not check the most part of these exact solutions. As the consequence of application of the Exp - function method we can usually obtain exact solutions which have to be simplified. However sometimes it is not easy to make simplifications taking into account cumbersome expressions. We are going to demonstrate that the main deficiency of the application of the Exp-function method is reducibility of the obtained exact solutions.

The paper is organized as follows. In section 2 we analyze the application of the Exp-function method to the Korteweg - de Vries - Burgers equation by Soliman [13]. We simplify exact solutions obtained by author and demonstrate that all his solutions are not new and were found before. In section 2 we consider the application of the Exp-function method to the Riccati equation by Zhang [14] and point out that these solutions are also simplified and these solutions are not new as well. In section 4 we analyze the work by Bekir and Boz [15] with application of the Exp-function method to the Sharma - Tisso - Olver equation and show that this exact solution can be simplified too. Section 5 of this paper is devoted to discussion of the application of the Exp-function method and other methods for finding exact solutions of nonlinear differential equations. In section 6 we introduce the modification of the simplest equation method and in section 7 we demonstrate the application of the modified simplest equation method to look for solution of the generalized Korteweg - de Vries equation.

2 Application of the Exp-function method to the Korteveg - de Vries - Burgers equation by Soliman

Let us analyze the application of the Exp-function method to search for exact solutions of the Burgers - Korteweg - de Vries - Burgers equation by Soliman [13]

$$u_t + \varepsilon\, u\, u_x + \mu\, u_{xxx} - \nu\, u_{xx} = 0 \tag{2.1}$$

Exact solutions of this equation in the form of the solitary wave at $\varepsilon = 1$ were first found in work [16] using the singular manifold method [17, 18]. It takes the form

$$u = C_0 + \frac{6\,\nu^2}{25\,\mu} - \frac{3\,\nu^2}{25\,\mu}\left(1 + \tanh\left\{\frac{\nu\, z}{10\,\mu}\right\}\right)^2, \qquad z = x - C_0\, t - x_0 \tag{2.2}$$

where C_0 and x_0 are arbitrary constants.

Solution (2.2) can be transformed to the following form

$$u = C_0 + \frac{6\,\nu^2}{25\,\mu} - \frac{12\,\nu^2}{25\,\mu\left(1 + e^{-\frac{2\,\nu\, z}{10\,\mu}}\right)^2}, \qquad z = x - C_0\, t - x_0 \tag{2.3}$$

Soliman in [13] obtained four "new exact solutions" of Eq. (2.1) using the Exp-function method. The first exact solution (solution (15) in [13]) was written in the form

$$u = \frac{\frac{4}{25}\left(\frac{25\varepsilon\,\mu\, a_{-1} - 3\,\nu^2\, b_0^2}{\varepsilon\,\mu\, b_0^2}\right) e^{\eta} + \frac{4\, a_{-1}}{b_0} + a_{-1}\, e^{-\eta}}{e^{\eta} + b_0 + \frac{b_0^2}{4}\, e^{-\eta}}, \tag{2.4}$$

$$\eta = k\,(x + \,\alpha\, t), \qquad \alpha = -\frac{2}{25}\left(\frac{50\,\varepsilon\,\mu\, a_{-1} - 3\,\nu^2\, b_0^2}{\mu\, b_0^2}\right)$$

Solution (2.4) of work [13] satisfies Eq. (2.1) in the case $k=\frac{\nu\,z}{10\,\mu}$. However exact solution (2.4) can be transformed taking into account the following set of equalities

$$u=\frac{\frac{4}{25}\left(\frac{25\varepsilon\,\mu\,a_{-1}-3\,\nu^2\,b_0^2}{\varepsilon\,\mu\,b_0^2}\right)\,e^{\eta}+\frac{4\,a_{-1}}{b_0}+a_{-1}\,e^{-\eta}}{e^{\eta}+b_0+\frac{b_0^2}{4}\,e^{-\eta}}=$$

$$=\frac{4\,a_{-1}}{b_0^2}-\frac{12\,\nu^2}{25\,\varepsilon\,\mu\,\left(1+\frac{b_0}{2}\,e^{-2\,\eta}\right)^2} \qquad (2.5)$$

Substituting η and α from (2.4) into (2.5) and taking into consideration $k=\frac{\nu\,z}{10\,\mu}$, $\frac{4\,a_{-1}}{b_0^2}=C_0-\frac{6\,\nu^2}{25\,\mu}$, $x_0=-\ln\frac{b_0}{2}$ and $\varepsilon=1$ we obtain solution (2.3). One can see that the solution (2.4) is transformed to solution (2.2) in the case $\varepsilon=1$ and (2.4) is not new solution.

It was also obtained the second "new exact solution" of the Korteweg - de Vries - Burgers equation in work [13] in the form

$$u=\frac{-\frac{3}{100}\left(\frac{16\nu^2\,b_1^3+225\,\varepsilon\,\mu\,a_{-1}}{\varepsilon\,\mu\,b_1^3}\right)\,e^{2\,\eta}-\frac{1}{100}\frac{-16\,\nu^2\,b_1^3+675\,\varepsilon\,\mu\,a_{-1}}{\varepsilon\,\mu\,b_1^2}\,e^{\eta}+a_{-1}\,e^{-\eta}}{e^{2\eta}+b_1\,e^{\eta}-\frac{4\,b_1^3}{27}\,e^{-\eta}},$$

$$\eta=k\,x+k\,\lambda\,t,\qquad \lambda=\frac{3}{100}\left(\frac{8\,\nu^2\,b_1^3+225\,\varepsilon\,\mu\,a_{-1}}{\mu\,b_1^3}\right) \qquad (2.6)$$

Solution (2.6) satisfies Eq. (2.1) again at $k=\frac{\nu}{10\,\mu}$. Solution (2.6) can be transformed as well by means of the following set of equalities

$$u=\frac{-\frac{3}{100}\left(\frac{16\nu^2\,b_1^3+225\,\varepsilon\,\mu\,a_{-1}}{\varepsilon\,\mu\,b_1^3}\right)\,e^{2\,\eta}-\frac{1}{100}\frac{-16\,\nu^2\,b_1^3+675\,\varepsilon\,\mu\,a_{-1}}{\varepsilon\,\mu\,b_1^2}\,e^{\eta}+a_{-1}\,e^{-\eta}}{e^{2\eta}+b_1\,e^{\eta}-\frac{4\,b_1^3}{27}\,e^{-\eta}}=$$

$$=\frac{-\frac{675\,a_{-1}}{100\,b_1^3}\left(e^{\eta}+\frac{2}{3}\,b_1\right)^2\left(e^{\eta}-\frac{b_1}{3}\right)-\frac{48\,\nu^2}{100\,\varepsilon\,\mu}\,e^{2\,\eta}\left(e^{\eta}-\frac{b_1}{3}\right)}{\left(e^{\eta}+\frac{2}{3}\,b_1\right)^2\left(e^{\eta}-\frac{b_1}{3}\right)}= \qquad (2.7)$$

$$=-\frac{27\,a_{-1}}{4\,b_1^3}-\frac{12\,\nu^2}{25\,\varepsilon\,\mu\,\left(1+\frac{2}{3}\,b_1\,e^{-2\,\eta}\right)}$$

The last expression can be transformed to the exact solution (2.2) of Eq. (2.1) if we assume $k = \frac{\nu z}{10 \mu}$, $-\frac{27 a_{-1}}{4 b_1^3} = C_0 - \frac{6 \nu^2}{25 \mu}$, $x_0 = -\ln \frac{2 b_1}{3}$ at $\varepsilon = 1$. We do not new exact solitary solution of the Korteweg - de Vries - Burgers equation again.

Two other exact solutions of work [13] (expressions (31) and (32)) do not satisfy the Korteweg - de Vries - Burgers equation except trivial case $k = 0$ and we do not study them.

We have the following results of our analysis. Using the Exp-function method Soliman in [13] obtained exact solutions in the cumbersome form and has not simplified these solutions. Unfortunately he has not found any new exact solutions of the Korteweg - de Vries - Burgers equation. His statement that "all the exact solutions of the KdV - Burgers equation are new" is not correct.

3 Application of the Exp-function method to the Riccati equation by Zhang

Using the Exp-function method Zhang presented "new generalized solitonary solutions of Riccati equation" in [14]. Let us illustrate that all solutions of Riccati equation are well known and can be included in the general solution of this equation.

Author in [14] has considered the Riccati equation in the form

$$\Phi_\xi = q + p\Phi^2 \tag{3.1}$$

The general solution of Eq. (3.1) can be written as [19, 20]

$$\Phi(\xi) = i\sqrt{\frac{q}{p}}\,\frac{1 + \exp(2i\sqrt{p\,q}\,(\xi - \xi_0))}{1 - \exp(2i\sqrt{p\,q}\,(\xi - \xi_0))} \tag{3.2}$$

It is well known that the Riccati equation can be transformed to the linear equation of the second order. Using transformation

$$\Phi(\xi) = -\frac{1}{p}\frac{\Psi_\xi}{\Psi}, \quad \Psi \equiv \Psi(\xi), \quad \Psi_\xi = \frac{d\Psi}{d\xi} \tag{3.3}$$

we have the linear equation from Eq. (3.1)

$$\Psi_{\xi\xi} + p\,q\,\Psi = 0 \tag{3.4}$$

General solution of Eq. (3.4) takes form

$$\Psi(\xi) = C_1 \exp(-i\sqrt{p\,q}\,\xi) + C_2 \exp(i\sqrt{p\,q}\,\xi) \tag{3.5}$$

Using (3.3) and (3.5) we get the general solution of Eq. (3.4) in the form

$$\Phi(\xi) = i\sqrt{\frac{q}{p}}\,\frac{C_1\exp(-i\sqrt{p\,q}\,\xi) - C_2\exp(i\sqrt{p\,q}\,\xi)}{C_1\exp(-i\sqrt{p\,q}\,\xi) + C_2\exp(i\sqrt{p\,q}\,\xi)} \tag{3.6}$$

Solution (3.6) can be transformed to solution (3.2) of Eq. (3.1) using the following equalities

$$\begin{gathered}\Phi(\xi) = i\sqrt{\frac{q}{p}}\,\frac{C_1\exp(-i\sqrt{p\,q}\,\xi) - C_2\exp(i\sqrt{p\,q}\,\xi)}{C_1\exp(-i\sqrt{p\,q}\,\xi) + C_2\exp(i\sqrt{p\,q}\,\xi)} = \\ = i\sqrt{\frac{q}{p}}\,\frac{1-\frac{C_2}{C_1}\exp(2i\sqrt{p\,q}\,\xi)}{1+\frac{C_2}{C_1}\exp(2i\sqrt{p\,q}\,\xi)} = i\sqrt{\frac{q}{p}}\,\frac{1+\exp(2i\sqrt{p\,q}\,(\xi-\xi_0))}{1-\exp(2i\sqrt{p\,q}\,(\xi-\xi_0))}\end{gathered} \tag{3.7}$$

where

$$\xi_0 = \frac{i}{2\sqrt{pq}} ln\left(-\frac{C_1}{C_2}\right) \tag{3.8}$$

It is clear that we can use general solution of Eq. (3.1) taking different forms into consideration but all these forms of solutions are the same solutions. We can write solution (3.2) and (3.6) using hyperbolic functions, trigonometric functions and so on. But we have to take into account that solution (3.2) is the general solution and other "new solutions" of equation (3.1) can not be found by any method.

Using the Exp-function method Zhang has found solution of Eq. (3.1) in [14] for the following three cases:

$$1)\,p = -\frac{1}{q}; \quad 2)\,p = \frac{1}{q}; \quad 3)\,p = -\frac{1}{4q}.$$

However all his solutions corresponds to expression (3.6). Let us demonstrate an example of reducible solution (Solution (27) in [14]) in the case $p = -\frac{1}{4q}$. Zhang has presented solution in the form

$$\Phi(\xi) = \frac{2b_1 q\exp(\xi+\omega) + a_0 + \frac{a_0^2-4b_0^2q^2}{8b_1 q}\exp(-\xi-\omega)}{b_1\exp(\xi+\omega) + b_0 - \frac{a_0^2-4b_0^2q^2}{16b_1 q^2}\exp(-\xi-\omega)} \tag{3.9}$$

However this solution can be transformed to (3.2). It follows from equalities

$$\Phi(\xi) = 2q\frac{16b_1^2q^2\exp(\xi+\omega) + 8a_0b_1q + (a_0^2 - 4b_0^2q^2)\exp(-\xi-\omega)}{16b_1^2q^2\exp(\xi+\omega) + 16b_0b_1q^2 - (a_0^2 - 4b_0^2q^2)\exp(-\xi-\omega)} =$$

$$= 2q\frac{(a_0 + 4b_1q\exp(\xi+\omega))^2 - 4b_0^2q^2}{4q^2(b_0 + 2b_1\exp(\xi+\omega))^2 - a_0^2} =$$

$$= 2q\frac{(a_0 + 4b_1q\exp(\xi+\omega) - 2b_0q)(a_0 + 4b_1q\exp(\xi+\omega) + 2b_0q)}{(2qb_0 + 4b_1q\exp(\xi+\omega) - a_0)(2qb_0 + 4b_1q\exp(\xi+\omega) + a_0)} =$$

$$= 2q\frac{4b_1q\exp(\xi+\omega) + a_0 - 2b_0q}{4b_1q\exp(\xi+\omega) - a_0 + 2b_0q} = 2q\frac{C_1\exp\xi - C_2}{C_1\exp\xi + C_2} \tag{3.10}$$

where

$$C_1 = 4b_1q\exp\omega; \quad C_2 = -a_0 + 2b_0q; \tag{3.11}$$

From the last expression we can see that solution (3.9) is not "new generalized solitonary solution" of the Riccati equation.

Solution (29) in [14] can be transformed as well. Zhang have presented exact solution of the Riccati equation (3.1) in the form

$$\Phi(\xi) = 2q\frac{\exp\xi + 2\sqrt{2q} + q\exp(-\xi)}{\exp\xi + 2\sqrt{q} - q\exp(-\xi)} \tag{3.12}$$

However this solution can be simplified to solution (3.6) of Eq. (3.1) at $p = -\frac{1}{4q}$ again by means of equalities

$$\Phi(\xi) = 2q\frac{\exp(2\xi) + 2\sqrt{2q}\exp\xi + 2q - q}{\exp(2\xi) + 2\sqrt{2q}\exp\xi + q - 2q} =$$

$$= 2q\frac{(\exp\xi + \sqrt{2q})^2 - q}{(\exp\xi + \sqrt{q})^2 - 2q} =$$

$$= 2q\frac{(\exp\xi + \sqrt{2q} - \sqrt{q})(\exp\xi + \sqrt{2q} + \sqrt{q})}{(\exp\xi + \sqrt{q} - \sqrt{2q})(\exp\xi + \sqrt{q} + \sqrt{2q})} = \tag{3.13}$$

$$= 2q\frac{\exp\xi + \sqrt{2q} - \sqrt{q}}{\exp\xi - \sqrt{2q} + \sqrt{q}} = 2q\frac{\exp\xi - C_2}{\exp\xi + C_2}$$

where

$$C_2 = \sqrt{q} + \sqrt{2q}; \tag{3.14}$$

So the Exp-function method does not allow Zhang in [14] to find out any new solution of the Riccati equation. We observe the same case as for the Burgers - Korteweg - de Vries equation by Soliman [13].

4 Application of the Exp-function method to the Klein - Gordon and to the Sharma - Tasso - Olver equations by Bekir and Boz

Bekir and Boz in [15] applied the Exp - function method to search for exact solutions of the Klein - Gordon, the Burgers - Fisher and the Sharma - Tasso - Olver equation.

For the Klein - Gordon equation

$$E_1[u] = u_{tt} - u_{xx} - u + u^3 = 0, \tag{4.1}$$

authors obtained the exact solutions of Eq. (4.1) in the form (formula (3.15) in [15])

$$u(x,t) = \frac{\exp(k\,x + \omega\,t) - \frac{1}{4}\,b_0^2\,\exp(-(kx + \omega\,t))}{\exp(k\,x + \omega\,t) + b_0 + \frac{1}{4}\,b_0^2\,\exp(-(kx + \omega\,t))} \tag{4.2}$$

Solution (4.2) can be transformed into simple form if we use the following equalities

$$u(x,t) = \frac{\exp(k\,x + \omega\,t) - \frac{1}{4}\,b_0^2\,\exp(-(kx + \omega\,t))}{\exp(k\,x + \omega\,t) + b_0 + \frac{1}{4}\,b_0^2\,\exp(-(kx + \omega\,t))} =$$

$$= \frac{\left(\exp\left(\frac{1}{2}(kx + \omega\,t)\right)\right)^2 - \left(\frac{b_0}{2}\exp\left(-\frac{1}{2}(kx + \omega\,t)\right)\right)^2}{\left(\exp\left(\frac{1}{2}(kx + \omega\,t) + \frac{b_0}{2}\exp\left(-\frac{1}{2}(kx + \omega\,t)\right)\right)^2} = \tag{4.3}$$

$$= \frac{\exp\left(\frac{1}{2}(k\,x + \omega\,t) - \frac{b_0}{2}\exp\left(-\frac{1}{2}(kx + \omega\,t)\right.\right.}{\exp\left(\frac{1}{2}(kx + \omega\,t) + \frac{b_0}{2}\exp\left(-\frac{1}{2}(kx + \omega\,t)\right.\right.}$$

The last solution can be easy found using the tanh-function method [21–23], the simple equation method [24–26] or the singular manifold method [17, 18, 27–34].

Bekir and Boz studied exact solutions of the Sharma - Tassa - Olver equation in [15]

$$u_t + \alpha\,\left(u^3\right)_x + \frac{3}{2}\,\alpha\,\left(u^2\right)_{xx} + \alpha\,u_{xxx} = 0 \tag{4.4}$$

Authors have looked for exact solutions of Eq. (4.4) using the traveling wave

$$u(x,t) = u(\xi), \qquad \xi = x - w\,t \tag{4.5}$$

Some solutions in [15] do not satisfy Eq. (4.4). However exact solution (5.18) in [15] in the form

$$u(x,t) = \frac{a_0 - k\,b_{-1}\,\exp\left(-(k\,x + \omega\,t)\right)}{\exp\left(k\,x + \omega\,t\right) - \frac{k^2\,b_{-1} + a_0^2}{k\,a_0} + b_{-1}\,\exp\left(-(k\,x + \omega\,t)\right)} \tag{4.6}$$

is correct but this exact solution can be transformed taking the following equalities into account

$$u(x,t) = \frac{a_0 - k\,b_{-1}\,\exp\left(-(k\,x + \omega\,t)\right)}{\exp\left(k\,x + \omega\,t\right) - \frac{k^2\,b_{-1} + a_0^2}{k\,a_0} + b_{-1}\,\exp\left(-(k\,x + \omega\,t)\right)} =$$

$$= \frac{a_0 - k\,b_{-1}\,\exp\left(-(k\,x + \omega\,t)\right)}{\left(\frac{1}{a_0}\,\exp\left(k\,x + \omega\,t\right) - \frac{1}{k}\right)\left(a_0 - k\,b_{-1}\,\exp\left(-(k\,x + \omega\,t)\right)\right)} = \tag{4.7}$$

$$= \frac{a_0\,k}{k\,\exp\left(k\,x + \omega\,t\right) - a_0}$$

Solution

$$u(x,t) = \frac{a_0\,k}{k\,\exp\left(k\,x + \omega\,t\right) - a_0} \tag{4.8}$$

satisfy Eq. (4.4) but this solution can be easy found using other methods.

More then that using the truncated expansion [16, 24, 29–34]

$$u(x,t) = \frac{F_x}{F}, \qquad F \equiv F(x,t) \tag{4.9}$$

we can transform the Sharma - Tasso - Olver equation (4.4) to the linear equation of the third order.

We obtain

$$E_3[u] = u_t + \alpha\ \left(u^3\right)_x + \frac{3}{2}\,\alpha\ \left(u^2\right)_{xx} + \alpha\, u_{xxx} =$$

$$= \frac{\partial}{\partial x}\left(\frac{F_t + \alpha\, F_{xxx}}{F}\right) = 0 \tag{4.10}$$

Using formula (4.9) and linear partial differential equation of the third order

$$F_t + \alpha\, F_{xxx} = 0 \tag{4.11}$$

we can have many exact solutions of the Sharma - Tasso - Olver equation (4.4).

5 Comparison of the Exp-function method with other methods

We know that all nonlinear partial differential equations can be separated on three types.

To the first type we can attribute all integrable partial differential equations. Partial differential equations of this type have the infinity amount of the exact solutions. The most known equations of this type are the Korteweg - de Vries equation, the Sine - Gordon equation, the nonlinear Schrodinger equation, the modified Korteweg - de Vries equation, the Boussinesq equation and the Kadomtsev - Petviashvili equation. This list can be continued but we believe that mentioned equations are basic integrable equations. The Cauchy problems for these equations can be solved using the inverse scattering transform [35–37]. Solitary wave solutions can be found for these equations taking the Hirota method into consideration [38].

The Exp-function method can be applied to these equations as well but we do not think that the Exp-function method is better than the Hirota method.

We can attribute the Burgers equation and other linearized differential equations to the first type as well. One can use the Cole - Hopf transformations [39, 40] and other transformations for these equations to obtain a lot of exact solutions .

Nonlinear partial differential equations without exact solutions belong to the second type of equations. There are a lot of examples of such equations but we give here only simple generalization of the Korteweg - de Vries equation in the form

$$u_t + 6\,u\,u_x + u_{xxx} + \alpha\,u = 0 \tag{5.1}$$

We have not got any method to look for exact solutions of such equations.

We can conclude all nonintegrable partial differential equations with some exact solutions to the third type of nonlinear differential equations. The Kuramoto - Sivashinsky equation, the Ginzburg - Landau equation, the Korteveg - De Vries - Burgers equation, the Fisher equation, the Fitzhugh - Nagumo equation and the Burgers - Huxley equation are the most known equations of this type. We have many different methods to search for exact solutions of such equations.

In last few decades great progress was made in the development of methods for finding exact solutions of nonlinear differential equations of the third type.

We can mention the singular manifold method [17,18,27–34], tanh-function method [21–23] and the simple equation method [24–26,41–50]. Certainly the mentioned methods can be applied to nonlinnear integrable differential equations as well but what for? We think the Exp-function method cedes to the enumerated methods because there are deficiencies which we demonstrated in sections 2 - 4.

From our point of view there is no single best method to search exact solutions of the nonlinear differential equations of the third type. Certainly each investigator of differential equations has his experience and his sympathy to methods but the choice of the method depends on form of the nonlinear differential equation and the pole of his solution.

One can think that there is the class of the nonlinear differential equations of the third type for effective application of the Exp-function method. This class has exact solutions with pole of the first order. The Fitzhugh - Nagumo equation, the Burgers - Huxley equation and some other equations [29, 30] can be included to this class of equations.

However we prefer to use the singular manifold method for such equations as well. Let us demonstrate this approach to look for exact solutions of the Fitzhugh - Nagumo equation [19, 29–31].

Let us take this equation in the form

$$u_t - u_{xx} + u(1-u)(\alpha - u) = 0 \tag{5.2}$$

Substituting

$$u = \sqrt{2}\,\frac{F_x}{F}, \qquad F = F(x,t) \tag{5.3}$$

into Eq. (5.2) and equating expressions at F^{-1} and F^{-2} to zero we have the following equations

$$F_{xt} + \alpha\, F_x - F_{xxx} = 0 \tag{5.4}$$

$$F_t - 3\, F_{xx} + \sqrt{2}\, (\alpha + 1)\, F_x = 0 \tag{5.5}$$

Solutions of this overdetermined system of equations can be easy found. Substituting F_t from (5.5) into Eq. (5.4) we have

$$2\, F_{xxx} - \sqrt{2}\, (\alpha + 1)\, F_{xx} + \alpha\, F_x = 0 \tag{5.6}$$

Solution of this equation can be presented in the form

$$F(x,t) = C_0(t) + C_1(t)\, e^{\lambda_1\, x} + C_2(t)\, e^{\lambda_2\, x}, \quad \alpha \neq 1 \tag{5.7}$$

where $\lambda_{1,2}$ are nonzero roots of equation

$$2\, \lambda^3 - \sqrt{2}\, (\alpha + 1)\, \lambda^2 + \alpha\, \lambda = 0 \tag{5.8}$$

Solving Eq. (5.8) we have

$$\lambda_0 = 0, \quad \lambda_1 = \frac{\sqrt{2}}{2}, \quad \lambda_2 = \frac{\alpha\, \sqrt{2}}{2}, \quad \alpha \neq 1 \tag{5.9}$$

and $F(x,t)$ in the form

$$F(x,t) = C_0(t) + C_1(t)\, e^{x\, \sqrt{2}/2} + C_2(t)\, e^{\alpha\, x\, \sqrt{2}/2} \tag{5.10}$$

Functions $C_0(t)$, $C_1(t)$ and $C_2(t)$ can be obtained after substitution (5.10) into Eq. (5.5). We have

$$F(x,t) = c_0 + c_1\, e^{(x\, \sqrt{2}/2 - \alpha\, t + t/2)} + c_2\, e^{\,(\alpha\, x \sqrt{2}/2 - \alpha\, t + \alpha^2\, t/2\,)}, \qquad \alpha \neq 1 \tag{5.11}$$

where c_0, c_1 and c_2 are arbitrary constants.

Substituting solutions for $F(x,t)$ into formula (5.3) for u we have exact solutions of the Fitzhugh - Nagumo equation in the form

$$u = \frac{c_1\, e^{(x\, \sqrt{2}/2 - \alpha\, t + t/2)} + c_2\, \alpha\, e^{\alpha\, /2\, \left(x \sqrt{2} + \alpha\, t - 2\, t\,\right)}}{c_0 + c_1\, e^{(x\, \sqrt{2}/2 - \alpha\, t + t/2)} + c_2\, e^{\alpha\, /2\, \left(x \sqrt{2} + \alpha\, t - 2\, t\,\right)}}, \qquad \alpha \neq 1 \tag{5.12}$$

At $\alpha = 1$ we can obtain $F(x,t)$ from the set of Eqs. (5.4) - (5.3) as well. It takes the form

$$F(x,t) = c_0 + (c_1 + c_2\, x + \sqrt{2}\, c_2\, t)\, e^{\,(x \sqrt{2}/2 - \, t/2\,)}, \tag{5.13}$$

Exact solution can be found from formula (5.3)

$$u=\frac{\sqrt{2}\,(c_1\,\sqrt{2}+2\,c_2+c_2\,\sqrt{2}\,x+2\,c_2\,t)\,e^{\,(x\sqrt{2}/2-\,t/2\,)}}{2\,\left(c_0+(c_1+c_2\,x+\sqrt{2}\,c_2\,t)\,e^{\,(x\sqrt{2}/2-\,t/2\,)}\right)},\tag{5.14}$$

The last solution can not be found by application of the Exp-function method. Exact solution (5.12) was not found by means of the Exp-function method as well but it could be found. However we think the application of the singular manifold approach for constructing exact solution to the Fitzhugh - Nagumo equation is easier than the application of the Exp-function method.

6 Modified simple equation method

The simple equation method is applied to find out an exact solution of a nonlinear ordinary differential equation

$$P(y,y',y'',y''',\dots)=0,\tag{6.1}$$

where $y=y(z)$ is an unknown function, P is a polynomial of the variable y and its derivatives.

To solve Eq. (6.1) we expand its solutions $y(z)$ in a finite series

$$y(z)=\sum_{k=0}^{N}A_kY^k,\quad A_k=\text{const},\quad A_N\neq 0,\tag{6.2}$$

where $Y=Y(z)$ are some special functions. These are, for example, the functions $\tanh(kz)$ for the tanh–method.

The basic idea of the simple equation method is the assumption that $Y=Y(z)$ are not only some special functions, but they are the functions that satisfy some ordinary differential equations. These ordinary differential equations are referred to as the simplest equations. Two main features characterize the simplest equation: first, this is the equation of a lesser order than Eq. (6.1); second, the general solution of this equation is known (or we know the way of finding its general solution). This means that the exact solutions $y(z)$ of Eq. (6.1) can be presented by a finite series (6.2) in the general solution $Y=Y(z)$ of the simplest equation.

One of the simple equation is the Riccati equation

$$Y'+Y^2+aY+b=0,\quad a,b=\text{const}.\tag{6.3}$$

The general solution of Eq. (6.3) is usually searched with the help of the standard anzats [19, 20]

$$Y=\frac{\psi'}{\psi},\tag{6.4}$$

where $\psi = \psi(z)$ is an unknown function to be found. This anzats leads to the second order linear ordinary differential equation

$$\psi'' + a\psi' + b\psi = 0 \tag{6.5}$$

that general solution $\psi = \psi(z)$ is well–known. Turning back to the variable Y (6.4) by applying the general solution $\psi = \psi(z)$ and substituting the ratio (6.4) in the expansion (6.2) we immediately obtain the exact solution $y = y(z)$ of Eq. (6.1).

Meanwhile, if one feels oneself ill at ease with differential equations then one can exclaim: How can I find this magic simplest equation? To get over this threatening obstacle we suggest to avoid this difficulty by writing the expansion (6.2) straight in the form

$$y(z) = \sum_{k=0}^{N} A_k Y^k = \sum_{k=0}^{N} A_k \left(\frac{\psi'}{\psi}\right)^k. \tag{6.6}$$

Therefore, the exact solutions $y = y(z)$ of the nonlinear ordinary differential equation (6.1) we could look for in the form

$$y(z) = \sum_{k=0}^{N} A_k \left(\frac{\psi'}{\psi}\right)^k, \quad A_k = \text{const}, \quad A_N \neq 0, \tag{6.7}$$

where the function $\psi = \psi(z)$ obeys Eq. (6.5).

Another simple equations are discussed in [28].

In the present paper we extend the simple equation method by the assumption that the function $\psi = \psi(z)$ is the general solution for the linear ordinary differential equation of the third order

$$\psi''' = \alpha\psi'' + \beta\psi' + \gamma\psi, \quad \alpha, \beta, \gamma = \text{const}. \tag{6.8}$$

Now we are going to find the exact solutions of some equations like Eq. (6.1) by using the extended simple equation method. This implies that we will search the solution $y = y(z)$ of Eq. (6.1) in a form of the expansion (6.7), where the function $\psi = \psi(z)$ obeys Eq. (6.8) and the coefficients A_k and the parameters α, β, γ are to be found.

Below we look round the main steps of our algorithm.

Firstly, for Eq. (6.1) we determine the positive number N in the expansion (6.7). To realise this procedure we concentrate our attention on the leading terms of Eq. (6.1). These are the terms that lead to the least positive p when a monomial $y = \frac{a}{z^p}$ is substituted in all the items of this equation. The homogeneous balance between the leading terms provides us the value of N. This value is also referred to the order of a pole for the solution of Eq. (6.1).

Secondly, we substitute in Eq. (6.1) the expansion (6.7) with the value of N already determined, we calculate all the necessary derivatives y', y'', y''', $\ldots$ of an unknown function $y = y(z)$ and we account that the function $\psi = \psi(z)$ satisfies Eq. (6.8). As a result of this substitution we get a polynomial with respect to the ratio $\frac{\psi'}{\psi}$ and its derivative $\left(\frac{\psi'}{\psi}\right)'$.

Thirdly, in the polynomial just obtained we gather the items with the same powers of the ratio $\frac{\psi'}{\psi}$ and its derivative $\left(\frac{\psi'}{\psi}\right)'$ and we equate with zero all the coefficients of this polynomial. This operation yields a system of algebraic equations with respect to the coefficients A_k of the expansion (6.7) and to the parameters α, β, γ of Eq. (6.8).

Fourthly, we solve the algebraic system.

Fifthly, in Eq. (6.8) we take the parameters α, β, γ that are the solutions of the algebraic system and derive the general solution $\psi = \psi(z)$ of Eq. (6.8) for them.

And finally, we substitute the general solution $\psi(z)$, its derivative $\psi'(z)$ and coefficients A_k in the expansion (6.7). The expansion (6.7) written in such form gives the exact solution of Eq. (6.1).

7 Application of the modified simplest equation method to generalization of the Korteweg - de Vries equation

Let us apply the modified simplest equation method to look for exact solutions of the generalized Korteweg–de Vries equation with source in the form

$$u_t + u_{xxx} - u_{xx} - uu_x + 3\,(u_x)^2 + 3uu_{xx} + 3u^2u_x + hu - 2u^2 + u^3 = 0, \quad (7.1)$$

where $u = u(x,t)$ is an unknown function, u_t, u_x, $\ldots$ are the partial derivatives of $u(x,t)$ and h is an arbitrary constant.

Using the travelling wave let us find exact solutions of equation

$$y''' + 3yy'' - y'' - yy' + 3\,(y')^2 - C_0y' + 3y^2y' + y^3 - 2y^2 + hy = 0. \quad (7.2)$$

We look for solutions of Eq. (7.2) in the form

$$y(z) = A_0 + A_1\,\frac{\psi_z}{\psi} \quad (7.3)$$

where $\psi(z)$ satisfies Eq. (6.8). Substituting (7.3) into Eq. (7.2) and taking Eq. (6.8) into account we obtain

$$A_1 = 1,$$

$$A_0 = \frac{2}{3} - \frac{\alpha}{3},$$

$$\beta = \frac{4}{3} - \frac{\alpha^2}{3} - h, \tag{7.4}$$

$$C_0 = 2 - h,$$

$$\delta = \frac{\alpha^3}{27} + \frac{16}{27} - \frac{4\,\alpha}{9} + \frac{h\alpha}{3} - \frac{2\,h}{3}$$

We have solution

$$y(z) = \frac{2}{3} - \frac{\alpha}{3} + \frac{\psi_z}{\psi} \tag{7.5}$$

Where $\psi(z)$ satisfy linear equation in the form

$$\psi_{zzz} - \alpha\psi_{zz} + \left(h + \frac{\alpha^2}{3} - \frac{4}{3}\right)\psi_z + \left(\frac{2h}{3} + \frac{4\alpha}{9} - \frac{h\alpha}{3} - \frac{\alpha^3}{27} - \frac{16}{27}\right)\psi = 0 \tag{7.6}$$

Solution $\psi(z)$ at $h < 1$ can be written in the form

$$\psi(z) = C_1 e^{\left(\frac{\alpha}{3} - \frac{2}{3}\right)z} + C_2 e^{\left(\frac{\alpha}{3} + \frac{1}{3} + \sqrt{1-h}\right)z} + C_3 e^{\left(\frac{\alpha}{3} + \frac{1}{3} - \sqrt{1-h}\right)z} \tag{7.7}$$

We have solution $y(z)$ of Eq. (7.2) at $C_0 = 2 - h$

$$y(z) = \frac{C_2\,\left(1 + \sqrt{1-h}\right) e^{z\left(1+\sqrt{1-h}\right)} + C_3\,\left(1 - \sqrt{1-h}\right)\,e^{-z\left(-1+\sqrt{1-h}\right)}}{C_1 + C_2 e^{z\left(1+\sqrt{1-h}\right)} + C_3 e^{-z\left(-1+\sqrt{1-h}\right)}} \tag{7.8}$$

Assuming $h = 1$ from Eq. (7.6) we have solution $\psi(z)$ in the form

$$\psi(z) = C_1 \mathrm{exp}\left(\frac{\alpha\,z}{3} - \frac{2\,z}{3}\right) + \left(C_2 + C_3\,z\right)\,\mathrm{exp}\left(\frac{\alpha\,z}{3} + \frac{z}{3}\right) \tag{7.9}$$

and solution of Eq. (7.2)

$$y(z) = \frac{C_2 + C_3(1+z)}{C_1\,\mathrm{exp}\,(-z) + C_2 + C_3 z} \tag{7.10}$$

Assuming $h = 1 + k^2$ in the case $h > 1$ we have solution $\psi(z)$

$$\psi(z) = C_1 e^{\left(\frac{\alpha z}{3} - \frac{2z}{3}\right)} + C_2 e^{\left(\frac{\alpha z}{3} + \frac{z}{3}\right)} \sin(k\,z) + C_3 e^{\left(\frac{\alpha z}{3} + \frac{z}{3}\right)} \cos(kz) \tag{7.11}$$

Exact solutions of Eq. (7.2) in this case takes the form

$$y(z) = \frac{(C_2\,k + C_3)\cos(k\,z) + (C_2 - C_3\,k)\sin(k\,z)}{C_1 \exp(-z) + C_2 \sin(k\,z) + C_3 \cos(k\,z)} \tag{7.12}$$

These solutions were not found by means of the Exp-function method.

8 Conclusion

We have given the analysis of the application of the Exp-function method for finding exact solutions of nonlinear differential equations. On the examples of papers [13–15] we have shown that this method allows us to search for exact solutions. However these exact solutions are cumbersome and as a rule we need to simplify them. Without simplifications one can think that we obtain "new solutions" of nonlinear differential equations.

We have discussed different methods for finding exact solutions. From our point of view we do not have the single best method to search for exact solutions of nonlinear nonintegrable differential equations. Sometimes we have to apply the singular manifold method [17, 18, 27–34], tanh-function method [21–23], the simple equation method [24–26], trial function method [51, 52] and so on.

In this paper we have presented the modified simple equation method and we think this method can be used to look for exact solutions in a number cases. We have illustrated our method to obtain exact solutions of the generalized Korteveg - de Vries equation with source.

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