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# Exact Solutions of Nonlinear Sets of Equations of the Theory of Heat and Mass Transfer in Reactive Media and Mathematical Biology

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**Abstract**—New classes of exact solutions of nonlinear sets of equations of the theory of heat and mass transfer in reactive media and mathematical biology are described. Various sets of the general form are considered, in which the chemical reaction rates depend on two or three arbitrary functions. Among the exact solutions obtained are solutions with ordinary, generalized, and functional separation of variables; time-periodic solutions; solutions that are periodic in the spatial coordinate; etc. A number of solutions contain arbitrary functions (they are expressed in terms of solutions of the linear heat equation and solutions of sets of ordinary differential equations). Sets describing the multicomponent reaction–diffusion and also some sets in several spatial variables are studied.

Exact solutions of nonlinear heat- and mass-transfer equations have always played an important role in the formation of a correct understanding of qualitative features of various processes in chemical engineering, thermophysics, and power engineering. Exact solutions of nonlinear equations vividly demonstrate and allow one to understand the mechanism of such complex nonlinear effects as the spatial localization of heat-transfer processes, the multiplicity or absence of steady states under certain conditions, the existence of blow-up modes, the presence or absence of periodic modes, etc. Simple solutions are widely used to illustrate the theoretical material, and some applications in lecture courses in universities and technical institutes (on heat- and mass-transfer theory, chemical engineering, hydrodynamics, gas dynamics, etc.).

Exact solutions of the traveling-wave type and self-similar solutions are often asymptotics of much wider classes of solutions under other initial and boundary conditions. This property allows one to draw general conclusions and predict the dynamics of various processes and phenomena. Even the partial exact solutions of nonlinear heat- and mass-transfer equations that have no clear physical meaning can be used as test problems for checking the validity and estimating the accuracy of various numerical, asymptotic, and approximate analytical methods. Moreover, model equations and problems that allow exact solutions serve as a basis for developing new numerical, asymptotic, and approximate methods, which, in their turn, enable one to study more complex problems that have no exact analytical solutions.

Note that many equations in chemical kinetics and chemical engineering contain empirical parameters or empirical functions. Exact solutions allow one to design experiment for determining these parameters or functions by artificially creating appropriate (boundary and initial) conditions. Of particular interest for design of experiments and testing of numerical and approximate methods are exact solutions of equations of the general form, which contain arbitrary functions. By exact solutions of heat- and mass-transfer equations and other nonlinear partial differential equations, the following solutions are usually meant:

(1) solutions that are expressed in terms of elementary functions or obtained by quadratures,

(2) solutions that are described by ordinary differential equations (sets of ordinary differential equations),

(3) solutions that are expressed in terms of solutions of linear partial differential equations (linear integral equations), and

(4) solutions that are described by sets of ordinary differential equations and linear partial differential equations.

Let us consider nonlinear sets of equations of the form

$$\frac{\partial u_m}{\partial t} = a_m \Delta u_m + F_m(u_1, \ldots, u_n), \quad m = 1, \ldots, n, \quad (1)$$

where $t$ is time, $\Delta$ is the Laplacian operator, and $F_m = F_m(u_1, \ldots, u_n)$ are kinetic functions. Depending on the problem being solved, the sought quantities can be concentrations $u_1, \ldots, u_n$ or concentrations $u_1, \ldots, u_{n-1}$ and temperature $u_n$. Such equations and sets of equations are widely used to mathematically describe various phenomena and processes in the theory of heat and mass transfer in reactive media [1–5], the theory of chemical reactors [6], and combustion theory [7] and also to mathematically model different processes in biology and biophysics [8–11].

Exact solutions of a single equation of form (1) for different kinetic functions $F = F(u)$ have been considered by many researchers [12–19]. Nikitin and Wiltshire [20, 21] and Cherniha and King [22, 23] presented a group classification of nonlinear sets of form (1) in two variables ($m = 2$), and Barannyk [24] gave some invariant and noninvariant exact solutions. In the general case, set (1) is invariant under shifts in independent variables (and under the substitution of $x$ for $-x$) and allows exact solutions of the traveling-wave type $u_m = u_m(z)$, where $z = kx + \lambda t$. Such solutions and also degenerate solutions, when one or more sought quantities are zero, are not considered here.

This paper provides a brief list of nonlinear sets of equations of the theory of heat and mass transfer in reactive media of form (1) with two, three, or more arbitrary functions that allow exact solutions. New classes of exact solutions with generalized and functional separation of variables and some other solutions are described.

Let us initially consider one-dimensional sets of two equations containing arbitrary functions $f(\varphi)$, $g(\varphi)$, and $h(\varphi)$ of argument $\varphi = \varphi(u_1, u_2)$; the equations are arranged in order of increasing complexity of this argument. For clarity, let us use the notation $u = u_1$, $w = u_2$.

## ARBITRARY FUNCTIONS ARE LINEAR FUNCTIONS OF THE SOUGHT QUANTITIES

**Set 1** contains two arbitrary functions, which are functions of one of the sought quantities:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + u\exp\left(k\frac{w}{u}\right)f(u),$$

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + \exp\left(k\frac{w}{u}\right)[wf(u) + g(u)].$$

The solution is [24]

$$u = y(\xi), \quad w = -\frac{2}{k}\ln|bx|y(\xi) + z(\xi),$$

$$\xi = \frac{x + C_3}{(C_1 t + C_2)^{1/2}},$$

where $C_1$, $C_2$, $C_3$, and $b$ are arbitrary constants and the functions $y = y(\xi)$ and $z = z(\xi)$ are found by solving the set of ordinary differential equations,

$$ay''_{\xi\xi} + \frac{1}{2}C_1\xi y'_\xi + \frac{1}{b^2\xi^2}y\exp\left(k\frac{z}{y}\right)f(y) = 0,$$

$$az''_{\xi\xi} + \frac{1}{2}C_1\xi z'_\xi - \frac{4a}{k\xi}y'_\xi + \frac{2a}{k\xi^2}y$$

$$+ \frac{1}{b^2\xi^2}\exp\left(k\frac{z}{y}\right)[zf(y) + g(y)] = 0.$$

**Set 2** contains three arbitrary functions, which are functions of a linear combination of the sought quantities:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf(bu - cw) + g(bu - cw),$$

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + wf(bu - cw) + h(bu - cw). \tag{2}$$

The particular case $f(z) = 0$, $g(z) = z$, and $h(z) = -z$ in set (2) describes a first-order reversible chemical reaction [2]. At $f(z) = z + k$ and $g(z) = h(z) = 0$, set (2) is a special case of the Lotka–Volterra model, which describes the competition between two biological species for the same food [9, pp. 35, 57]. In mathematical biophysics, set (2) at $f(z) = z + k_1$, $g(z) = k_2 z$, and $h(z) = k_3 z$ [9, p. 37] is also encountered.

1°. The solution is

$$u = \varphi(t) + c\exp\left[\int f(b\varphi - c\psi)dt\right]\theta(x, t),$$

$$w = \psi(t) + b\exp\left[\int f(b\varphi - c\psi)dt\right]\theta(x, t),$$

where $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are found by solving the set of ordinary differential equations

$$\varphi'_t = \varphi f(b\varphi - c\psi) + g(b\varphi - c\psi),$$

$$\psi'_t = \psi f(b\varphi - c\psi) + h(b\varphi - c\psi),$$

and the function $\theta = \theta(x, t)$ obeys the linear heat equation

$$\frac{\partial\theta}{\partial t} = \frac{a\partial^2\theta}{\partial x^2}.$$

2°. Let us multiply the first and second equations of set (2) by $b$ and $-c$, respectively, and add the products obtained. This leads to the equation

$$\frac{\partial\zeta}{\partial t} = a\frac{\partial^2\zeta}{\partial x^2} + \zeta f(\zeta) + bg(\zeta) - ch(\zeta),$$

$$\zeta = bu - cw, \tag{3}$$

which is considered here together with the first equation of the initial set:

$$\frac{\partial u}{\partial t} = \frac{a\partial^2 u}{\partial x^2} + uf(\zeta) + g(\zeta). \tag{4}$$

Equation (3) can be studied separately. An extensive list of exact solutions of equations of this form for different kinetic functions $F(\zeta) = \zeta f(\zeta) + bg(\zeta) - ch(\zeta)$ is available in the literature [17, 19] (for a given function $F$, two of the three functions $f$, $g$, and $h$ can be specified arbitrarily). Note that, in the general case, Eq. (3) allows an exact solution of the traveling-wave type $\zeta = \zeta(z)$, where $z = kx - \lambda t$.

If a certain solution $\zeta = = \zeta(x, t)$ of Eq. (3) is known, then the function $u = u(x, t)$ can be determined by solving linear Eq. (4) and the function $w = w(x, t)$ is found from the expression $w = (bu - \zeta)/c$.

**Set 3** contains two arbitrary functions, which are functions of a linear combination of the sought quantities and exponential functions:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + e^{\lambda u} f(\lambda u - \sigma w),$$

$$\frac{\partial w}{\partial t} = b\frac{\partial^2 w}{\partial x^2} + e^{\sigma w} g(\lambda u - \sigma w).$$

1°. The solution is

$$u = y(\xi) - \frac{1}{\lambda}\ln(C_1 t + C_2),$$

$$w = z(\xi) - \frac{1}{\sigma}\ln(C_1 t + C_2), \quad \xi = \frac{x + C_3}{(C_1 t + C_2)^{1/2}},$$

where $C_1$, $C_2$, and $C_3$ are arbitrary constants and the functions $y = y(\xi)$ and $z = z(\xi)$ are found by solving the set of ordinary differential equations

$$a y''_{\xi\xi} + \frac{1}{2}C_1 \xi y'_{\xi} + \frac{C_1}{\lambda} + e^{\lambda y} f(\lambda y - \sigma z) = 0,$$

$$b z''_{\xi\xi} + \frac{1}{2}C_1 \xi z'_{\xi} + \frac{C_1}{\sigma} + e^{\sigma z} g(\lambda y - \sigma z) = 0.$$

2°. The solution at $b = a$ is

$$u = \theta(x, t), \quad w = \frac{\lambda}{\sigma}\theta(x, t) - \frac{k}{\sigma},$$

where $k$ is the root of the algebraic (transcendental) equation

$$\lambda f(k) = \sigma e^{-k} g(k),$$

and the function $\theta = \theta(x, t)$ obeys the differential equation

$$\frac{\partial \theta}{\partial t} = \frac{a\partial^2 \theta}{\partial x^2} + f(k)e^{\lambda\theta}.$$

For exact solutions of this equation, see [17, 19].

## ARBITRARY FUNCTIONS ARE FUNCTIONS OF THE RATIO OF THE SOUGHT QUANTITIES

**Set 4** contains two arbitrary functions, which are functions of the ratio of the sought quantities:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf\left(\frac{u}{w}\right), \quad \frac{\partial w}{\partial t} = b\frac{\partial^2 w}{\partial x^2} + wg\left(\frac{u}{w}\right). \quad (5)$$

The particular case $f(z) = k_1 - k_2 z^{-1}$, $g(z) = k_2 - k_1 z$ characterizes a first-order reversible chemical reaction [2]. The Eigen–Schuster model [11] (see also [9, pp. 31–32]), which describes the competition between populations for the food substrate at constant multiplication coefficients, leads to set (5) at $f(z) = k/(z + 1)$ and $g(z) = -kz/(z + 1)$, where $k$ is the difference of the multiplication coefficients.

1°. The solution in the form of the product of functions of different arguments, which is periodic in the spatial coordinate, is

$$u = [C_1 \sin(kx) + C_2 \cos(kx)]\varphi(t),$$

$$w = [C_1 \sin(kx) + C_2 \cos(kx)]\psi(t),$$

where $C_1$, $C_2$, and $k$ are arbitrary constants and the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are found by solving the set of ordinary differential equations

$$\varphi'_t = -ak^2\varphi + \varphi f(\varphi/\psi),$$

$$\psi'_t = -bk^2\psi + \psi g(\varphi/\psi).$$

2°. The solution in the form of the product of functions of different arguments is

$$u = [C_1 \exp(kx) + C_2 \exp(-kx)]U(t),$$

$$w = [C_1 \exp(kx) + C_2 \exp(-kx)]W(t),$$

where $C_1$, $C_2$, and $k$ are arbitrary constants and the functions $U = U(t)$ and $W = W(t)$ are found by solving the set of ordinary differential equations

$$U'_t = ak^2 U + Uf(U/W),$$

$$W'_t = bk^2 W + Wg(U/W).$$

3°. The degenerate solution in the form of the product of functions of different arguments is

$$u = (C_1 x + C_2)U(t), \quad w = (C_1 x + C_2)W(t),$$

where $C_1$ and $C_2$ are arbitrary constants and the functions $U = U(t)$ and $W = W(t)$ are found by solving the set of ordinary differential equations

$$U'_t = Uf(U/W), \quad W'_t = Wg(U/W).$$

This autonomous set can be integrated, since, after elimination of the variable $t$, it is reduced to a first-order homogeneous equation (similarly, the corresponding sets from examples 1° and 2° are integrated).

4°. The solution, in the form of the product of functions of different arguments (which rapidly decays as $t \longrightarrow \infty$ at $\lambda > 0$), is

$$u = e^{-\lambda t} y(x), \quad w = e^{-\lambda t} z(x),$$

where $\lambda$ is an arbitrary constant and the functions $y = y(x)$ and $z = z(x)$ are found by solving the set of ordinary differential equations

$$a y''_{xx} + \lambda y + yf(y/z) = 0,$$

$$b z''_{xx} + \lambda z + zg(y/z) = 0.$$

5°. The solution in the form of the product of two traveling waves with different velocities (which generalizes the solution from example 4°) is

$$u = e^{kx-\lambda t}y(\xi), \quad w = e^{kx-\lambda t}z(\xi), \quad \xi = \beta x - \gamma t,$$

where $k$, $\lambda$, $\beta$, and $\gamma$ are arbitrary constants and the functions $y = y(\xi)$ and $z = z(\xi)$ are found by solving the set of ordinary differential equations

$$a\beta^2 y''_{\xi\xi} + (2ak\beta + \gamma)y'_\xi + (ak^2 + \lambda)y + yf(y/z) = 0,$$
$$b\beta^2 z''_{\xi\xi} + (2bk\beta + \gamma)z'_\xi + (bk^2 + \lambda)z + zg(y/z) = 0.$$

The particular case $k = \lambda = 0$ describes a solution of the traveling-wave type. At $k = \gamma = 0$ and $\beta = 1$, the solution coincides with the solution from example 4°.

**Set 4a** contains two arbitrary functions, which are functions of the ratio of the sought quantities:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf\left(\frac{u}{w}\right), \quad \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + wg\left(\frac{u}{w}\right). \quad (6)$$

This set is a particular case of set (5) at $b = a$ and allows the above solutions from examples 1°–5°. Moreover, set (5) has interesting properties and other solutions, which are presented below.

6°. The solution of the point-source type is

$$u = \exp\left(-\frac{x^2}{4at}\right)\varphi(t), \quad w = \exp\left(-\frac{x^2}{4at}\right)\psi(t),$$

where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are found by solving the set of ordinary differential equations

$$\varphi'_t = -\frac{1}{2t}\varphi + \varphi f\left(\frac{\varphi}{\psi}\right), \quad \psi'_t = -\frac{1}{2t}\psi + \psi g\left(\frac{\varphi}{\psi}\right).$$

7°. The solution with functional separation of variables is

$$u = \exp\left(kxt + \frac{2}{3}ak^2t^3 - \lambda t\right)y(\xi), \quad \xi = x + akt^2,$$
$$w = \exp\left(kxt + \frac{2}{3}ak^2t^3 - \lambda t\right)z(\xi),$$

where $k$ and $\lambda$ are arbitrary constants and the functions $y = y(\xi)$ and $z = z(\xi)$ are found by solving the set of ordinary differential equations

$$ay''_{\xi\xi} + (\lambda - k\xi)y + yf(y/z) = 0,$$
$$az''_{\xi\xi} + (\lambda - k\xi)z + zg(y/z) = 0.$$

8°. Let $k$ be the root of the algebraic (transcendental) equation

$$f(k) = g(k). \quad (7)$$

Then, there is an exact solution of the form

$$u = ke^{\lambda t}\theta, \quad w = e^{\lambda t}\theta, \quad \lambda = f(k),$$

where the function $\theta = \theta(x, t)$ obeys the linear heat equation

$$\frac{\partial\theta}{\partial t} = \frac{a\partial^2\theta}{\partial x^2}. \quad (8)$$

9°. The solution that is periodic in time $t$ is

$$u = Ak\exp(-\mu x)\sin(\beta x - 2a\beta\mu t + B),$$
$$w = A\exp(-\mu x)\sin(\beta x - 2a\beta\mu t + B),$$
$$\beta = \left(\mu^2 + \frac{1}{a}f(k)\right)^{1/2},$$

where $A$, $B$, and $\mu$ are arbitrary constants and $k$ is the root of algebraic (transcendental) equation (7).

10°. The solution

$$u = \varphi(t)\exp\left[\int g(\varphi(t))dt\right]\theta(x, t),$$
$$w = \exp\left[\int g(\varphi(t))dt\right]\theta(x, t),$$

where the function $\varphi = \varphi(t)$ is described by a nonlinear first-order ordinary differential equation with separable variables:

$$\varphi'_t = [f(\varphi) - g(\varphi)]\varphi, \quad (9)$$

and the function $\theta = \theta(x, t)$ obeys linear heat equation (8). The partial solution $\varphi = k = \text{const}$ of Eq. (9) corresponds to the solution form example 8°. The general solution of Eq. (9) is written in the implicit form

$$\int\frac{d\varphi}{[f(\varphi) - g(\varphi)]\varphi} = t + C.$$

**Note.** Let $u = u(x, t)$ and $w = w(x, t)$ be a solution of set (6). Then, the functions

$$u_1 = Au(\pm x + C_1, t + C_2),$$
$$w_1 = Aw(\pm x + C_1, t + C_2);$$
$$u_2 = \exp(\lambda x + a\lambda^2 t)u(x + 2a\lambda t, t),$$
$$w_2 = \exp(\lambda x + a\lambda^2 t)w(x + 2a\lambda t, t),$$

(where $A$, $C_1$, $C_2$, and $\lambda$ are arbitrary constants) are also solutions of the same equations. These expressions enable one to "breed" exact solutions. For this purpose, for example, one can use the solutions of set (5) at $b = a$, which were presented above in examples 1°–5°, and the solution from examples 6°, 7°.

**Set 5** contains three arbitrary functions, which are functions of the ratio of the sought quantities:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf\left(\frac{u}{w}\right) + g\left(\frac{u}{w}\right),$$
$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + wf\left(\frac{u}{w}\right) + h\left(\frac{u}{w}\right).$$

Let $k$ be the root of the algebraic (transcendental) equation

$$g(k) = kh(k).$$

1°. The solution at $f(k) \neq 0$ is

$$u(x, t) = k\left(\exp[f(k)t]\theta(x, t) - \frac{h(k)}{f(k)}\right),$$

$$w(x, t) = \exp[f(k)t]\theta(x, t) - \frac{h(k)}{f(k)},$$

where the function $\theta = \theta(x, t)$ obeys linear heat equation (8).

2°. The solution at $f(k) = 0$ is

$$u(x, t) = k[\theta(x, t) + h(k)t],$$
$$w(x, t) = \theta(x, t) + h(k)t,$$

where the function $\theta = \theta(x, t)$ obeys linear heat equation (8).

**Set 6** contains three arbitrary functions, which are functions of the ratio of the sought quantities:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf\left(\frac{u}{w}\right) + \frac{u}{w}h\left(\frac{u}{w}\right),$$

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + wg\left(\frac{u}{w}\right) + h\left(\frac{u}{w}\right).$$

The solution is

$$u = \varphi(t)G(t)\left[\theta(x, t) + \int\frac{h(\varphi)}{G(t)}dt\right],$$

$$w = G(t)\left[\theta(x, t) + \int\frac{h(\varphi)}{G(t)}dt\right],$$

$$G(t) = \exp\left[\int g(\varphi)dt\right],$$

where the function $\varphi = \varphi(t)$ is described by nonlinear first-order ordinary differential equation (9) with separable variables and the function $\theta = \theta(x, t)$ obeys linear heat equation (8).

**Set 7** contains two arbitrary functions, which are functions of the ratio of the sought quantities:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + u^3 f\left(\frac{u}{w}\right), \quad \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + u^3 g\left(\frac{u}{w}\right).$$

1°. The noninvariant solution is [24, 25]

$$u = (x + A)\varphi(z), \quad w = (x + A)\psi(z),$$

$$z = t + \frac{1}{6a}(x + A)^2 + B,$$

where $A$ and $B$ are arbitrary constants and the functions $\varphi = \varphi(z)$ and $\psi = \psi(z)$ are found by solving the set of ordinary differential equations

$$\varphi''_{zz} + 9a\varphi^3 f(\varphi/\psi) = 0,$$
$$\psi''_{zz} + 9a\varphi^3 g(\varphi/\psi) = 0.$$

2°. Self-similar solutions: see set 9, example 1° at $n = 3$.

**Set 8** contains two arbitrary functions, which are functions of the ratio of the sought quantities:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + au - u^3 f\left(\frac{u}{w}\right),$$

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + aw - u^3 g\left(\frac{u}{w}\right).$$

1°. The solution at $a > 0$ is [24]

$$u = \left[C_1\exp\left(\frac{1}{2}\sqrt{2a}x + \frac{3}{2}at\right) - C_2\exp\left(-\frac{1}{2}\sqrt{2a}x + \frac{3}{2}at\right)\right]\varphi(z),$$

$$w = \left[C_1\exp\left(\frac{1}{2}\sqrt{2a}x + \frac{3}{2}at\right) - C_2\exp\left(-\frac{1}{2}\sqrt{2a}x + \frac{3}{2}at\right)\right]\psi(z),$$

$$z = C_1\exp\left(\frac{1}{2}\sqrt{2a}x + \frac{3}{2}at\right) + C_2\exp\left(-\frac{1}{2}\sqrt{2a}x + \frac{3}{2}at\right) + C_3,$$

where $C_1$, $C_2$, and $C_3$ are arbitrary constants and the functions $\varphi = \varphi(z)$ and $\psi = \psi(z)$ are found by solving the set of ordinary differential equations

$$a\varphi''_{zz} = 2\varphi^3 f(\varphi/\psi), \quad a\psi''_{zz} = 2\varphi^3 g(\varphi/\psi).$$

2°. The solution at $a < 0$ is [24]

$$u = \exp\left(\frac{3}{2}at\right)\sin\left(\frac{1}{2}\sqrt{2|a|}x + C_1\right)U(\xi),$$

$$w = \exp\left(\frac{3}{2}at\right)\sin\left(\frac{1}{2}\sqrt{2|a|}x + C_1\right)W(\xi),$$

$$\xi = \exp\left(\frac{3}{2}at\right)\cos\left(\frac{1}{2}\sqrt{2|a|}x + C_1\right) + C_2,$$

where $C_1$ and $C_2$ are arbitrary constants and the functions $U = U(\xi)$ and $W = W(\xi)$ are found by solving the set of ordinary differential equations

$$aU''_{\xi\xi} = -2U^3 f(U/W), \quad aW''_{\xi\xi} = -2U^3 g(U/W).$$

**Set 9** contains two arbitrary functions, which are functions of the ratio of the sought quantities, and a power-law function:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + u^n f\left(\frac{u}{w}\right), \quad \frac{\partial w}{\partial t} = b\frac{\partial^2 w}{\partial x^2} + w^n g\left(\frac{u}{w}\right).$$

At $f(z) = kz^{-m}$ and $g(z) = -kz^{n-m}$, this set describes a chemical reaction of order $n$ (of order $n - m$ in the component $u$ and of order $m$ in the component $w$); the values $n = 2$ and $m = 1$ characterize the frequently encountered case of a second-order reaction [1, 2, 4].

1°. The self-similar solution is

$$u = (C_1 t + C_2)^{1/(1-n)} y(\xi),$$

$$w = (C_1 t + C_2)^{1/(1-n)} z(\xi), \quad \xi = \frac{x + C_3}{(C_1 t + C_2)^{1/2}},$$

where $C_1$, $C_2$, and $C_3$ are arbitrary constants and the functions $y = y(\xi)$ and $z = z(\xi)$ are found by solving the set of ordinary differential equations

$$a y''_{\xi\xi} + \frac{1}{2} C_1 \xi y'_\xi + \frac{C_1}{n-1} y + y^n f\left(\frac{y}{z}\right) = 0,$$

$$b z''_{\xi\xi} + \frac{1}{2} C_1 \xi z'_\xi + \frac{C_1}{n-1} z + z^n g\left(\frac{y}{z}\right) = 0.$$

2°. The solution at $b = a$ is

$$u(x, t) = k\theta(x, t), \quad w(x, t) = \theta(x, t),$$

where $k$ is the root of the algebraic (transcendental) equation

$$k^{n-1} f(k) = g(k),$$

and the function $\theta = \theta(x, t)$ obeys an equation containing a power-law function:

$$\frac{\partial \theta}{\partial t} = \frac{a \partial^2 \theta}{\partial x^2} + g(k)\theta^n.$$

For exact solutions of this equation, see [14, 19, 24].

**Set 10** contains three arbitrary functions, which are functions of the ratio of the sought quantities, and logarithmic functions:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf\left(\frac{u}{w}\right)\ln u + ug\left(\frac{u}{w}\right),$$

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + wf\left(\frac{u}{w}\right)\ln w + wh\left(\frac{u}{w}\right).$$

The solution is

$$u(x, t) = \varphi(t)\psi(t)\theta(x, t), \quad w(x, t) = \psi(t)\theta(x, t),$$

where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are found by solving the first-order ordinary differential equations

$$\varphi'_t = \varphi[g(\varphi) - h(\varphi) + f(\varphi)\ln\varphi], \quad (10)$$

$$\psi'_t = \psi[h(\varphi) + f(\varphi)\ln\psi], \quad (11)$$

and the function $\theta = \theta(x, t)$ is described by the differential equation

$$\frac{\partial \theta}{\partial t} = a\frac{\partial^2 \theta}{\partial x^2} + f(\varphi)\theta\ln\theta. \quad (12)$$

The solution of Eq. (10) with separable variables can be represented in implicit form. Equation (11) is easy to integrate, since the change of variable $\psi = e^\zeta$ reduces it to a linear equation. Equation (12) allows exact solutions of the form

$$\theta = \exp[\sigma_2(t)x^2 + \sigma_1(t)x + \sigma_0(t)],$$

where the functions $\sigma_n(t)$ are described by the equations

$$\sigma'_2 = f(\varphi)\sigma_2 + 4a\sigma_2^2,$$

$$\sigma'_1 = f(\varphi)\sigma_1 + 4a\sigma_1\sigma_2,$$

$$\sigma'_0 = f(\varphi)\sigma_0 + a\sigma_1^2 + 2a\sigma_2.$$

This set can be sequentially integrated, since the first equation is the Bernoulli equation and the second and third equations are linear in the sought function. Note that the first equation has the partial solution $\sigma_2 = 0$.

**Note.** Equation (10) allows the singular solution $\varphi = k = \text{const}$, where $k$ is the root of the algebraic (transcendental) equation $g(k) - h(k) + f(k)\ln k = 0$.

## ARBITRARY FUNCTIONS ARE FUNCTIONS OF THE PRODUCT OF VARIOUS POWERS OF THE SOUGHT QUANTITIES

**Set 11** contains two arbitrary functions, which are functions of the product of various powers of the sought quantities:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf(u^n w^m),$$

$$\frac{\partial w}{\partial t} = b\frac{\partial^2 w}{\partial x^2} + wg(u^n w^m).$$

The solution in the form of the product of several traveling waves with different velocities:

$$u = e^{m(kx - \lambda t)} y(\xi), \quad w = e^{-n(kx - \lambda t)} z(\xi),$$
$$\xi = \beta x - \gamma t,$$

where $k$, $\lambda$, $\beta$, and $\gamma$ are arbitrary constants and the functions $y = y(\xi)$ and $z = z(\xi)$ are found by solving the set of ordinary differential equations

$$a\beta^2 y''_{\xi\xi} + (2akm\beta + \gamma)y'_\xi + m(ak^2 m + \lambda)y + yf(y^n z^m) = 0,$$

$$b\beta^2 z''_{\xi\xi} + (-2bkn\beta + \gamma)z'_\xi + n(bk^2 n - \lambda)z + zg(y^n z^m) = 0.$$

The particular case $k = \lambda = 0$ describes a solution of the traveling-wave type.

**Note.** The solution obtained generalizes the results of Barannyk [24], who considered the case $\lambda = 0$.

**Set 12** contains two arbitrary functions, which are functions of the product of various powers of the sought quantities, and power-law functions:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + u^{1+kn} f(u^n w^m),$$

$$\frac{\partial w}{\partial t} = b\frac{\partial^2 w}{\partial x^2} + w^{1-km} g(u^n w^m).$$

The self-similar solution is

$$u = (C_1 t + C_2)^{-1/(kn)} y(\xi), \quad w = (C_1 t + C_2)^{1/(km)} z(\xi),$$

$$\xi = \frac{x + C_3}{(C_1 t + C_2)^{1/2}},$$

where $C_1$, $C_2$, and $C_3$ are arbitrary constants and the functions $y = y(\xi)$ and $z = z(\xi)$ are found by solving the set of ordinary differential equations

$$a y''_{\xi\xi} + \frac{1}{2} C_1 \xi y'_\xi + \frac{C_1}{kn} y + y^{1+kn} f(y^n z^m) = 0,$$

$$b z''_{\xi\xi} + \frac{1}{2} C_1 \xi z'_\xi - \frac{C_1}{km} z + z^{1-km} g(y^n z^m) = 0.$$

**Set 13** contains two arbitrary functions, which are functions of the product of various powers of the sought quantities, and logarithmic functions:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + cu\ln u + uf(u^n w^m),$$

$$\frac{\partial w}{\partial t} = b\frac{\partial^2 w}{\partial x^2} + cw\ln w + wg(u^n w^m).$$

1°. The solution is

$$u = \exp(Ame^{ct}) y(\xi), \quad w = \exp(-Ane^{ct}) z(\xi),$$

$$\xi = kx - \lambda t,$$

where $A$, $k$, and $\lambda$ are arbitrary constants and the functions $y = y(\xi)$ and $z = z(\xi)$ are found by solving the set of ordinary differential equations

$$ak^2 y''_{\xi\xi} + \lambda y'_\xi + cy\ln y + yf(y^n z^m) = 0,$$

$$bk^2 z''_{\xi\xi} + \lambda z'_\xi + cz\ln z + zg(y^n z^m) = 0.$$

The particular case $A = 0$ describes a solution of the traveling-wave type. At $\lambda = 0$, there is a solution in the form of the product of two functions of time $t$ and coordinate $x$.

2°. At $a = b$ and $n = -m$, see set (10), in which one should take $f(u/w) = c$ and, then, sequentially redenote the function $g$ by $f$ and the function $h$ by $g$.

ARBITRARY FUNCTIONS ARE FUNCTIONS OF THE SUM OR DIFFERENCE OF THE SQUARES OF THE SOUGHT QUANTITIES

**Set 14** contains two arbitrary functions, which are functions of the sum of the squares of the sought quantities:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf(u^2 + w^2) - wg(u^2 + w^2),$$

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + wf(u^2 + w^2) + ug(u^2 + w^2).$$

At $f(z) = k - z$ and $g(z) = -2$, the set under consideration without diffusion ($a = 0$) was used to model kinematic waves in Belousov–Zhabotinskii-type reactions [8, p. 211].

1°. The solution that is periodic in the spatial coordinate with phase shift of the components:

$$u = \psi(t)\cos\varphi(x, t), \quad w = \psi(t)\sin\varphi(x, t),$$

$$\varphi(x, t) = C_1 x + \int g(\psi^2)dt + C_2,$$

Here, $C_1$ and $C_2$ are arbitrary constants and the function $\psi = \psi(t)$ is found from the autonomous ordinary differential equation

$$\psi'_t = \psi f(\psi^2) - aC_1^2\psi,$$

the general solution of which can be written in the implicit form

$$\int \frac{d\psi}{\psi f(\psi^2) - aC_1^2\psi} = t + C_3.$$

2°. The solution that is periodic in time with phase shift of the components:

$$u = r(x)\cos[\theta(x) + C_1 t + C_2],$$

$$w = r(x)\sin[\theta(x) + C_1 t + C_2],$$

where $C_1$ and $C_2$ are arbitrary constants and the functions $r = r(x)$ and $\theta = \theta(x)$ are found by solving the set of ordinary differential equations

$$ar''_{xx} - ar(\theta'_x)^2 + rf(r^2) = 0,$$

$$ar\theta''_{xx} + 2ar'_x\theta'_x - C_1 r + rg(r^2) = 0.$$

3°. The solution (which generalizes the solution from example 2°) is [24]

$$u = r(z)\cos[\theta(z) + C_1 t + C_2],$$

$$w = r(z)\sin[\theta(z) + C_1 t + C_2], \quad z = x + \lambda t,$$

where $C_1$, $C_2$, and $\lambda$ are arbitrary constants and the functions $r = r(z)$ and $\theta = \theta(z)$ are found by solving the set of ordinary differential equations

$$ar''_{zz} - ar(\theta'_z)^2 - \lambda r'_z + rf(r^2) = 0,$$

$$ar\theta''_{zz} + 2ar'_z\theta'_z - \lambda r\theta'_z - C_1 r + rg(r^2) = 0.$$

**Note.** The set considered and its solutions can be obtained by separating the real and imaginary parts of the generalized Landau–Ginzburg equation of special form [19, p. 135, Eq. (3) at $b = 0$].

**Set 15** contains two arbitrary functions, which are functions of the difference of the squares of the sought quantities:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf(u^2 - w^2) + wg(u^2 - w^2),$$

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + wf(u^2 - w^2) + ug(u^2 - w^2).$$

1°. The solution is

$$u = \psi(t)\cosh\varphi(x, t), \quad w = \psi(t)\sinh\varphi(x, t),$$

$$\varphi(x, t) = C_1 x + \int g(\psi^2)dt + C_2,$$

Here, $C_1$ and $C_2$ are arbitrary constants and the function $\psi = \psi(t)$ is found from the autonomous ordinary differential equation

$$\psi'_t = \psi f(\psi^2) + aC_1^2\psi,$$

the general solution of which can be written in the implicit form

$$\int \frac{d\psi}{\psi f(\psi^2) + aC_1^2\psi} = t + C_3.$$

2°. The solution is

$$u = r(x)\cosh[\theta(x) + C_1 t + C_2],$$

$$w = r(x)\sinh[\theta(x) + C_1 t + C_2],$$

where $C_1$ and $C_2$ are arbitrary constants and the functions $r = r(x)$ and $\theta = \theta(x)$ are found by solving the set of ordinary differential equations

$$ar''_{xx} + ar(\theta'_x)^2 + rf(r^2) = 0,$$

$$ar\theta''_{xx} + 2ar'_x\theta'_x + rg(r^2) - C_1 r = 0.$$

3°. The solution (which generalizes the solution from example 2°) is

$$u = r(z)\cosh[\theta(z) + C_1 t + C_2],$$

$$w = r(z)\sinh[\theta(z) + C_1 t + C_2], \quad z = x + \lambda t,$$

where $C_1$, $C_2$, and $\lambda$ are arbitrary constants and the functions $r = r(z)$ and $\theta = \theta(z)$ are found by solving the set of ordinary differential equations

$$ar''_{zz} + ar(\theta'_z)^2 - \lambda r'_z + rf(r^2) = 0,$$

$$ar\theta''_{zz} + 2ar'_z\theta'_z - \lambda r\theta'_z - C_1 r + rg(r^2) = 0.$$

**Set 16** contains three arbitrary functions, which are functions of the sum of the squares of the sought quantities, and inverse trigonometric functions:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf(u^2 + w^2) - wg(u^2 + w^2) - w\arctan\left(\frac{w}{u}\right)h(u^2 + w^2),$$

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + wf(u^2 + w^2) + ug(u^2 + w^2) + u\arctan\left(\frac{w}{u}\right)h(u^2 + w^2).$$

The solution with functional separation of variables (which, at fixed $t$, determines the structure that is periodic in $x$ with phase shifts of the components) is

$$u = r(t)\cos[\varphi(t)x + \psi(t)],$$

$$w = r(t)\sin[\varphi(t)x + \psi(t)],$$

where the functions $r = r(t)$, $\varphi = \varphi(t)$, and $\psi = \psi(t)$ are found by solving the set of ordinary differential equations

$$r'_t = -ar\varphi^2 + rf(r^2),$$

$$\varphi'_t = h(r^2)\varphi,$$

$$\psi'_t = h(r^2)\psi + g(r^2).$$

**Set 17** contains three arbitrary functions, which are functions of the difference of the squares of the sought quantities, and inverse hyperbolic functions:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf(u^2 - w^2) + wg(u^2 - w^2) + w\,\mathrm{arctanh}\left(\frac{w}{u}\right)h(u^2 - w^2),$$

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + wf(u^2 - w^2) + ug(u^2 - w^2) + u\,\mathrm{arctanh}\left(\frac{w}{u}\right)h(u^2 - w^2).$$

The solution with functional separation of variables is

$$u = r(t)\cosh[\varphi(t)x + \psi(t)],$$

$$w = r(t)\sinh[\varphi(t)x + \psi(t)],$$

where the functions $r = r(t)$, $\varphi = \varphi(t)$, and $\psi = \psi(t)$ are found by solving the set of ordinary differential equations

$$r'_t = ar\varphi^2 + rf(r^2),$$

$$\varphi'_t = h(r^2)\varphi,$$

$$\psi'_t = h(r^2)\psi + g(r^2).$$

## ARBITRARY FUNCTIONS ARE FUNCTIONS OF A COMPLEX COMBINATION OF THE SOUGHT QUANTITIES

**Set 18** contains two arbitrary functions, which are functions of a complex argument:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + u^{k+1}f(\varphi), \quad \varphi = u\exp\left(-\frac{w}{u}\right),$$

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + u^{k+1}[f(\varphi)\ln u + g(\varphi)].$$

The solution is [24]

$$u = (C_1 t + C_2)^{-1/k} y(\xi), \quad \xi = \frac{x + C_3}{(C_1 t + C_2)^{1/2}},$$

$$w = (C_1 t + C_2)^{-1/k}\left[z(\xi) - \frac{1}{k}\ln(C_1 t + C_2) y(\xi)\right],$$

where $C_1$, $C_2$, and $C_3$ are arbitrary constants and the functions $y = y(\xi)$ and $z = z(\xi)$ are found by solving the set of ordinary differential equations

$$ay''_{\xi\xi} + \frac{1}{2}C_1\xi y'_\xi + \frac{C_1}{k}y + y^{k+1}f(\varphi) = 0,$$

$$bz''_{\xi\xi} + \frac{1}{2}C_1\xi z'_\xi + \frac{C_1}{k}z + \frac{C_1}{k}y$$

$$+ y^{k+1}[f(\varphi)\ln y + g(\varphi)] = 0,$$

$$\varphi = y\exp\left(-\frac{z}{y}\right).$$

**Set 19** contains two arbitrary functions, which are functions of different arguments:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf(u^2 + w^2) - wg\left(\frac{w}{u}\right),$$

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + wf(u^2 + w^2) + ug\left(\frac{w}{u}\right).$$

The solution with functional separation of variables is

$$u = r(x, t)\cos\varphi(t), \quad w = r(x, t)\sin\varphi(t),$$

where the function $\varphi = \varphi(t)$ obeys the first-order ordinary differential equation with separable variables,

$$\varphi'_t = g(\tan\varphi), \tag{13}$$

and the function $r = r(x, t)$ is described by the differential equation

$$\frac{\partial r}{\partial t} = \frac{a\partial^2 r}{\partial x^2} + rf(r^2). \tag{14}$$

The general solution of Eq. (13) is expressed in the implicit form

$$\int\frac{d\varphi}{g(\tan\varphi)} = t + C.$$

Equation (14) allows an exact solution of the traveling-wave type $r = r(z)$, where $z = kx - \lambda t$ ($k$ and $\lambda$ are arbitrary constants) and the function $r(z)$ is determined by the autonomous ordinary differential equation

$$ak^2 r''_{zz} + \lambda r'_z + rf(r^2) = 0.$$

For exact solutions of Eq. (14) at different functions $f$, see [17, 19].

**Set 20** contains two arbitrary functions, which are functions of different arguments:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + uf(u^2 - w^2) + wg\left(\frac{w}{u}\right),$$

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + wf(u^2 - w^2) + ug\left(\frac{w}{u}\right).$$

The solution with functional separation of variables is

$$u = r(x, t)\cosh\varphi(t), \quad w = r(x, t)\sinh\varphi(t),$$

where the function $\varphi = \varphi(t)$ obeys the first-order ordinary differential equation with separable variables:

$$\varphi'_t = g(\tanh\varphi),$$

and the function $r = r(x, t)$ is described by differential equation (14) (about its solutions, see set (19)).

## STEADY-STATE SETS IN TWO INDEPENDENT VARIABLES

**Set 21** contains two arbitrary functions, which are functions of the ratio of the sought quantities:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = uf\left(\frac{u}{w}\right), \quad \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = wg\left(\frac{u}{w}\right).$$

1°. The solution in the form of the product of functions of different arguments that is periodic in the spatial variable $x$ (a similar solution can be obtained by interchanging $x$ and $y$) is

$$u = [C_1\sin(kx) + C_2\cos(kx)]\varphi(y),$$

$$w = [C_1\sin(kx) + C_2\cos(kx)]\psi(y),$$

where $C_1$, $C_2$, and $k$ are arbitrary constants and the functions $\varphi = \varphi(y)$ and $\psi = \psi(y)$ are found by solving the set of ordinary differential equations

$$\varphi''_{yy} = k^2\varphi + \varphi f(\varphi/\psi),$$

$$\psi''_{yy} = k^2\psi + \psi g(\varphi/\psi).$$

2°. The solution in the form of the product of functions of different arguments is

$$u = [C_1\exp(kx) + C_2\exp(-kx)]U(y),$$

$$w = [C_1\exp(kx) + C_2\exp(-kx)]W(y),$$

where $C_1$, $C_2$, and $k$ are arbitrary constants and the functions $U = U(y)$ and $W = W(y)$ are found by solving the set of ordinary differential equations

$$U''_{yy} = -k^2 U + Uf(U/W),$$
$$W''_{yy} = -k^2 W + Wg(U/W).$$

3°. The degenerate solution in the form of the product of functions of different arguments is

$$u = (C_1 x + C_2)U(y), \quad w = (C_1 x + C_2)W(y),$$

where $C_1$ and $C_2$ are arbitrary constants and the functions $U = U(y)$ and $W = W(y)$ are found by solving the set of ordinary differential equations

$$U''_{yy} = Uf(U/W), \quad W''_{yy} = Wg(U/W).$$

4°. The solution in the form of the product of functions of different arguments is

$$u = e^{a_1 x + b_1 y}\xi(z), \quad w = e^{a_1 x + b_1 y}\eta(z),$$
$$z = a_2 x + b_2 y,$$

where $a_1$, $a_2$, $b_1$, and $b_2$ are arbitrary constants and the functions $\xi = \xi(z)$ and $\eta = \eta(z)$ are found by solving the set of ordinary differential equations

$$(a_2^2 + b_2^2)\xi''_{zz} + 2(a_1 a_2 + b_1 b_2)\xi'_z$$
$$+ (a_1^2 + b_1^2)\xi = \xi f(\xi/\eta),$$
$$(a_2^2 + b_2^2)\eta''_{zz} + 2(a_1 a_2 + b_1 b_2)\eta'_z$$
$$+ (a_1^2 + b_1^2)\eta = \eta g(\xi/\eta).$$

**Note.** Sets 21–23 have axisymmetric solutions $u = u(r)$ and $w = w(r)$, where $r = (x^2 + y^2)^{1/2}$, which are not considered here.

**Set 22** contains two arbitrary functions, which are functions of the product of various powers of the sought quantities:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = uf(u^n w^m),$$
$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = wg(u^n w^m).$$

The solution in the form of the product of functions of different arguments:

$$u = e^{m(a_1 x + b_1 y)}\xi(z), \quad w = e^{-n(a_1 x + b_1 y)}\eta(z),$$
$$z = a_2 x + b_2 y,$$

where $a_1$, $a_2$, $b_1$, and $b_2$ are arbitrary constants and the functions $\xi = \xi(z)$ and $\eta = \eta(z)$ are found by solving the set of ordinary differential equations

$$(a_2^2 + b_2^2)\xi''_{zz} + 2m(a_1 a_2 + b_1 b_2)\xi'_z$$
$$+ m^2(a_1^2 + b_1^2)\xi = \xi f(\xi^n \eta^m),$$
$$(a_2^2 + b_2^2)\eta''_{zz} - 2n(a_1 a_2 + b_1 b_2)\eta'_z$$
$$+ n^2(a_1^2 + b_1^2)\eta = \eta g(\xi^n \eta^m).$$

**Set 23** contains two arbitrary functions, which are functions of the sum of the squares of the sought quantities:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = uf(u^2 + w^2) - wg(u^2 + w^2),$$
$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = wf(u^2 + w^2) + ug(u^2 + w^2).$$

1°. The solution that is periodic in the $y$ coordinate with phase shift of the components:

$$u = r(x)\cos[\theta(x) + C_1 y + C_2],$$
$$w = r(x)\sin[\theta(x) + C_1 y + C_2],$$

where $C_1$ and $C_2$ are arbitrary constants and the functions $r = r(x)$ and $\theta = \theta(x)$ are found by solving the set of ordinary differential equations

$$r''_{xx} = r(\theta'_x)^2 + C_1^2 r + rf(r^2),$$
$$r\theta''_{xx} = -2r'_x\theta'_x + rg(r^2).$$

2°. The solution (which generalizes the solution from example 1°) is

$$u = r(z)\cos[\theta(z) + C_1 y + C_2],$$
$$w = r(z)\sin[\theta(z) + C_1 y + C_2],$$
$$z = k_1 x + k_2 y,$$

where $C_1$, $C_2$, $k_1$, and $k_2$ are arbitrary constants and the functions $r = r(z)$ and $\theta = \theta(z)$ are found by solving the set of ordinary differential equations

$$(k_1^2 + k_2^2)r''_{zz} = k_1^2 r(\theta'_z)^2 + r(k_2\theta'_z + C_1)^2 + rf(r^2),$$
$$(k_1^2 + k_2^2)r\theta''_{zz} = -2[(k_1^2 + k_2^2)\theta'_x + C_1 k_2]r'_x + rg(r^2).$$

## SOME GENERALIZATIONS FOR SETS OF TWO EQUATIONS

A number of the above results allow essential generalizations.

**Set 24** contains an arbitrary operator and three arbitrary functions of two arguments, which are functions of a linear combination of the sought quantities and time (set 24 generalizes set 2):

$$\frac{\partial u}{\partial t} = L[u] + uf(t, bu - cw) + g(t, bu - cw),$$

$$\frac{\partial w}{\partial t} = L[w] + wf(t, bu - cw) + h(t, bu - cw).$$

Here, $L$ is an arbitrary linear differential operator in the spatial variables $x_1, \ldots, x_n$ (of any order in the derivatives), the coefficients of which may be functions of $x_1, \ldots, x_n, t$.

Below are examples of operators that are frequently used in heat- and mass-transfer theory:

$$L[u] = \frac{\partial}{\partial x}\left[a(x)\frac{\partial u}{\partial x}\right] + \beta(x)\frac{\partial u}{\partial x},$$

$$L[u] = a\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}\right).$$

The first operator is used in one-dimensional problems of convective mass transfer with variable diffusion coefficient, and the second in various three-dimensional problems.

The solution is

$$u = \varphi(t) + c\exp\left[\int f(t, b\varphi - c\psi)dt\right]\theta(x, t),$$

$$w = \psi(t) + b\exp\left[\int f(t, b\varphi - c\psi)dt\right]\theta(x, t),$$

where $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are found by solving the set of ordinary differential equations

$$\varphi'_t = \varphi f(t, b\varphi - c\psi) + g(t, b\varphi - c\psi),$$

$$\psi'_t = \psi f(t, b\varphi - c\psi) + h(t, b\varphi - c\psi),$$

and the function $\theta = \theta(x_1, \ldots, x_n, t)$ obeys the linear equation

$$\frac{\partial \theta}{\partial t} = L[\theta]. \quad (15)$$

**Set 25** contains an arbitrary operator and two arbitrary functions of two arguments, which are functions of the ratio of the sought quantities and time (set 25 generalizes set 4):

$$\frac{\partial u}{\partial t} = L[u] + uf\left(t, \frac{u}{w}\right),$$

$$\frac{\partial w}{\partial t} = L[w] + wg\left(t, \frac{u}{w}\right).$$

Here, $L$ is an arbitrary linear differential operator described in set 24.

The solution is

$$u = \varphi(t)\exp\left[\int g(t, \varphi(t))dt\right]\theta(x_1, \ldots, x_n, t),$$

$$w = \exp\left[\int g(t, \varphi(t))dt\right]\theta(x_1, \ldots, x_n, t),$$

where the function $\varphi = \varphi(t)$ is described by the nonlinear first-order ordinary differential equation

$$\varphi'_t = [f(t, \varphi) - g(t, \varphi)]\varphi, \quad (16)$$

and the function $\theta = \theta(x_1, \ldots, x_n, t)$ obeys linear equation (15).

**Note.** The coefficients of the operator $L$ may also be functions of the ratio $u/w$ of the sought functions. As an example, let us consider a nonlinear set of the reaction–diffusion type:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[f\left(t, \frac{u}{w}\right)\frac{\partial u}{\partial x}\right] + ug\left(t, \frac{u}{w}\right),$$

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[f\left(t, \frac{u}{w}\right)\frac{\partial w}{\partial x}\right] + wh\left(t, \frac{u}{w}\right),$$

which contains three arbitrary functions of two arguments. This set has an exact solution of the form

$$u = \varphi(t)\exp\left[\int h(t, \varphi(t))dt\right]\theta(x, t),$$

$$w = \exp\left[\int h(t, \varphi(t))dt\right]\theta(x, t),$$

where the function $\varphi = \varphi(t)$ is described by the ordinary differential equation

$$\varphi'_t = [g(t, \varphi) - h(t, \varphi)]\varphi,$$

and the function $\theta = \theta(x, t)$ obeys the linear equation

$$\frac{\partial \theta}{\partial t} = f(t, \varphi(t))\frac{\partial^2 \theta}{\partial x^2}. \quad (17)$$

The substitution $\tau = \int f(t, \varphi(t))dt$ reduces Eq. (17) to the linear heat equation

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial x^2}.$$

**Set 26** contains an arbitrary operator and three arbitrary functions of two arguments, which are functions of the ratio of the sought quantities and time (set 26 generalizes set 6):

$$\frac{\partial u}{\partial t} = L[u] + uf\left(t, \frac{u}{w}\right) + \frac{u}{w}h\left(t, \frac{u}{w}\right),$$

$$\frac{\partial w}{\partial t} = L[w] + wg\left(t, \frac{u}{w}\right) + h\left(t, \frac{u}{w}\right).$$

Here, $L$ is an arbitrary linear differential operator described in set 24.

The solution is

$$u = \varphi(t)G(t)\left[\theta(x_1, \ldots, x_n, t) + \int\frac{h(t,\varphi)}{G(t)}dt\right],$$

$$w = G(t)\left[\theta(x_1, \ldots, x_n, t) + \int\frac{h(t,\varphi)}{G(t)}dt\right],$$

$$G(t) = \exp\left[\int g(t,\varphi)dt\right],$$

where the function $\varphi = \varphi(t)$ is described by nonlinear first-order ordinary differential equation (16) and the function $\theta = \theta(x_1, \ldots, x_n, t)$ obeys linear equation (15).

**Set 27** contains two arbitrary operators and two arbitrary functions of two arguments, which are functions of the ratio of the sought quantities and time (set 27 generalizes set 4):

$$\partial u/\partial t = L_1[u] + uf(u/w),$$
$$\partial w/\partial t = L_2[w] + wg(u/w).$$

Here, $L_1$ and $L_2$ are arbitrary linear differential operators (of any order) in the $x$ coordinate with constant coefficients.

1°. The solution in the form of the product of two traveling waves with different velocities is

$$u = e^{kx-\lambda t}y(\xi), \quad w = e^{kx-\lambda t}z(\xi), \quad \xi = \beta x - \gamma t,$$

where $k$, $\lambda$, $\beta$, and $\gamma$ are arbitrary constants and the functions $y = y(\xi)$ and $z = z(\xi)$ are found by solving the set of ordinary differential equations

$$M_1[y] + \lambda y + yf(y/z) = 0,$$
$$M_2[z] + \lambda z + zg(y/z) = 0;$$

$$M_1[y] = e^{-kx}L_1[e^{kx}y(\xi)],$$
$$M_2[z] = e^{-kx}L_2[e^{kx}z(\xi)].$$

The particular case $k = \lambda = 0$ describes a solution of the traveling-wave type.

2°. If the operators contain only even derivatives, there are solutions of the form

$$u = [C_1\sin(kx) + C_2\cos(kx)]\varphi(t),$$
$$w = [C_1\sin(kx) + C_2\cos(kx)]\psi(t);$$
$$u = [C_1\exp(kx) + C_2\exp(-kx)]\varphi(t),$$
$$w = [C_1\exp(kx) + C_2\exp(-kx)]\psi(t);$$
$$u = (C_1x + C_2)\varphi(t),$$
$$w = (C_1x + C_2)\psi(t),$$

where $C_1$, $C_2$, and $k$ are arbitrary constants (the first solution is periodic in the spatial coordinate, and the third solution is degenerate). Note that the coefficients of the operators $L_1$ and $L_2$ and the functions $f$ and $g$ may also be functions of time.

**Set 28** contains an arbitrary operator and three arbitrary functions of two arguments, which are functions of the ratio of the sought quantities and time (set 28 generalizes set 10):

$$\frac{\partial u}{\partial t} = L[u] + uf\left(t, \frac{u}{w}\right)\ln u + ug\left(t, \frac{u}{w}\right),$$

$$\frac{\partial w}{\partial t} = L[w] + wf\left(t, \frac{u}{w}\right)\ln w + wh\left(t, \frac{u}{w}\right).$$

Here, $L$ is an arbitrary linear differential operator in the spatial variables $x_1, \ldots, x_n$ (of any order in the derivatives), the coefficients of which may be functions of $x_1, \ldots, x_n, t$.

The solution is

$$u = \varphi(t)\psi(t)\theta(x_1, \ldots, x_n, t),$$
$$w = \psi(t)\theta(x_1, \ldots, x_n, t),$$

where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are found by solving the set of ordinary differential equations

$$\begin{aligned}\varphi'_t &= \varphi[g(t,\varphi) - h(t,\varphi) + f(t,\varphi)\ln\varphi],\\ \psi'_t &= \psi[h(t,\varphi) + f(t,\varphi)\ln\psi],\end{aligned} \qquad (18)$$

and the function $\theta = \theta(x_1, \ldots, x_n, t)$ obeys the differential equation

$$\frac{\partial\theta}{\partial t} = L[\theta] + f(t,\varphi)\theta\ln\theta. \qquad (19)$$

If a solution of the first equation of set (18) is known, then a solution of the second equation can be found by the change of variable $\psi = e^{\zeta}$ (it is reduced to a linear equation for $\zeta$). If $L$ is a one-dimensional operator ($n = 1$) with constant coefficients and the functions $f$ = const then Eq. (19) has a solution of the traveling-wave type $\theta = \theta(kx - \lambda t)$.

## SOME GENERALIZATIONS FOR MULTICOMPONENT SETS

**Set 29.** Let us consider the multicomponent set

$$\begin{aligned}\partial u_m/\partial t &= L[u_m]\\ &\quad + u_m f(t, u_1 - b_1u_n, \ldots, u_{n-1} - b_{n-1}u_n)\\ &\quad + g_m(t, u_1 - b_1u_n, \ldots, u_{n-1} - b_{n-1}u_n),\\ m &= 1, \ldots, n,\end{aligned}$$

which contains $n + 1$ arbitrary functions $f, g_1, \ldots, g_n$ of $n$ arguments. Here, $L$ is an arbitrary linear differential operator in the spatial variables $x_1, \ldots, x_n$ (of any order in the derivatives), whose coefficients may be functions of $x_1, \ldots, x_n, t$. It is supposed that $L[\text{const}] = 0$.

The solution is

$$\begin{aligned}u_m &= \varphi_m(t)\\ &\quad + \exp\left[\int f(t, \varphi_1 - b_1\varphi_n, \ldots, \varphi_{n-1} - b_{n-1}\varphi_n)dt\right]\\ &\quad \times\theta(x_1, \ldots, x_n, t).\end{aligned}$$

Here, the functions $\varphi_m = \varphi_m(t)$ are found by solving the set of ordinary differential equations

$$\varphi'_t = \varphi_m f(t, \varphi_1 - b_1\varphi_n, \ldots, \varphi_{n-1} - b_{n-1}\varphi_n) + g_m(t, \varphi_1 - b_1\varphi_n, \ldots, \varphi_{n-1} - b_{n-1}\varphi_n),$$

where $m = 1, \ldots, n$; the prime means differentiation with respect to $t$; and the function $\theta = \theta(x_1, \ldots, x_n, t)$ obeys the linear equation

$$\frac{\partial\theta}{\partial t} = L[\theta]. \qquad (20)$$

**Set 30.** Let us consider the multicomponent set ($m = 1, \ldots, n-1$)

$$\frac{\partial u_m}{\partial t} = L[u_m] + u_m f_m\left(t, \frac{u_1}{u_n}, \ldots, \frac{u_{n-1}}{u_n}\right) + \frac{u_m}{u_n} g\left(t, \frac{u_1}{u_n}, \ldots, \frac{u_{n-1}}{u_n}\right),$$

$$\frac{\partial u_n}{\partial t} = L[u_n] + u_n f_n\left(t, \frac{u_1}{u_n}, \ldots, \frac{u_{n-1}}{u_n}\right) + g\left(t, \frac{u_1}{u_n}, \ldots, \frac{u_{n-1}}{u_n}\right),$$

which contains $n + 1$ arbitrary functions $f_1, \ldots, f_n, g$ of $n$ arguments. Here, $L$ is an arbitrary linear differential operator described in set 29.

The solution is

$$u_m = \varphi_m(t)F_n(t)\left[\theta(x_1, \ldots, x_n, t) + \int \frac{g(t, \varphi_1, \ldots, \varphi_{n-1})}{F_n(t)} dt\right], \quad m = 1, \ldots, n-1,$$

$$u_n = F_n(t)\left[\theta(x_1, \ldots, x_n, t) + \int \frac{g(t, \varphi_1, \ldots, \varphi_{n-1})}{F_n(t)} dt\right],$$

$$F_n(t) = \exp\left[\int f_n(t, \varphi_1, \ldots, \varphi_{n-1}) dt\right],$$

where the functions $\varphi_m = \varphi_m(t)$ are found by solving the set of ordinary differential equations

$$\varphi'_m = \varphi_m[f_m(t, \varphi_1, \ldots, \varphi_{n-1}) - f_n(t, \varphi_1, \ldots, \varphi_{n-1})], \quad m = 1, \ldots, n-1,$$

and the function $\theta = \theta(x_1, \ldots, x_n, t)$ obeys linear equation (20).

**Some comments.** The sets of the general form (containing arbitrary functions) and their exact solutions, which were considered above, enable one to model various heat- and mass-transfer processes in stagnant and flowing reactive media. The results obtained can be used to analyze nonlinear effects and develop new models in mathematical biology and chemical engineering, and also to design some experiments. The exact solutions presented allow one to efficiently test various numerical and approximate methods.

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