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New Ansätze and Exact Solutions for Nonlinear Reaction-Diffusion Equations Arising in Mathematical Biology

Roman M. CHERNIHA

Institute of Mathematics of the National Academy of Sciences of Ukraine,
3 Tereshchenkivs'ka Str., Kyiv 4, Ukraine
E-mail: chern@apmat.freenet.kiev.ua

Abstract

The method of additional generating conditions is applied for finding new non-Lie ansätze and exact solutions of nonlinear generalizations of the Fisher equation.

1. Introduction

In the present paper, I consider nonlinear reaction-diffusion equations with convection term of the form

$$U_t = [A(U)U_x]_x + B(U)U_x + C(U), \tag{1}$$

where $U = U(t, x)$ is an unknown function, $A(U), B(U), C(U)$ are arbitrary smooth functions. The indices t and x denote differentiating with respect to these variables. Equation (1) generalizes a great number of the well-known nonlinear second-order evolution equations describing various processes in biology [1]–[3].

Equation (1) contains as a particular case the classical Burgers equation

$$U_t = U_{xx} + \lambda_1 U U_x \tag{2}$$

and the well-known Fisher equation [4]

$$U_t = U_{xx} + \lambda_2 U - \lambda_3 U^2, \tag{3}$$

where λ_1, λ_2, and $\lambda_3 \in \mathbf{R}$. A particular case of equation (1) is also the Murray equation [1]–[2]

$$U_t = U_{xx} + \lambda_1 U U_x + \lambda_2 U - \lambda_3 U^2, \tag{4}$$

which can be considered as a generalization of the Fisher and Burgers equations.

Construction of particular exact solutions for nonlinear equations of the form (1) remains an important problem. Finding exact solutions that have a biological interpretation is of fundamental importance.

On the other hand, the well-known principle of linear superposition cannot be applied to generate new exact solutions to *nonlinear* partial differential equations (PDEs). Thus, the classical methods are not applicable for solving nonlinear partial differential equations.

Of course, a change of variables can sometimes be found that transforms a nonlinear partial differential equation into a linear equation, but finding exact solutions of most nonlinear partial differential equations generally requires new methods.

Now, the very popular method for construction of exact solutions to nonlinear PDEs is the Lie method [5, 6]. However it is well known that some very popular nonlinear PDEs have a poor Lie symmetry. For example, the Fisher equation (3) and the Murray equation (4) are invariant only under the time and space translations. The Lie method is not efficient for such PDEs since in these cases it enables us to construct ansätze and exact solutions, which can be obtained without using this cumbersome method.

A constructive method for obtaining non-Lie solutions of nonlinear PDEs and a system of PDEs has been suggested in [7, 8]. The method (see Section 2) is based on the consideration of a fixed nonlinear PDE (a system of PDEs) together with an *additional generating condition* in the form of a linear high-order ODE (a system of ODEs). Using this method, new exact solutions are obtained for nonlinear equations of the form (1) (Section 3). These solutions are applied for solving some nonlinear boundary-value problems.

2. A constructive method for finding new exact solutions of nonlinear evolution equations

Here, the above-mentioned method to the construction of exact solutions is briefly presented. Consider the following class of nonlinear evolution second-order PDEs

$$U_t = (\lambda + \lambda_0 U)U_{xx} + rU_x^2 + pUU_x + qU^2 + sU + s_0, \tag{5}$$

where coefficients $\lambda_0, \lambda, r, p, q, s$, and s_0 are arbitrary constants or arbitrary smooth functions of t. It is easily seen that the class of PDEs (1) contains this equation as a particular case. On the other hand equation (5) is a generalization of the known nonlinear equations (2)–(4).

If coefficients in (5) are constants, then this equation is invariant with respect to the transformations

$$x' = x + x_0 \qquad t' = t + t_0, \tag{6}$$

and one can find plane wave solutions of the form

$$U = U(kx + vt), \quad v, k \in \mathbf{R}. \tag{7}$$

But here, such solutions are not constructed since great number papers are devoted to the construction of plane wave solutions for nonlinear PDEs of the form (1) and (5) (see references in [9], for instance).

Hereinafter, I consider (5) together with the *additional generating conditions* in the form of linear high-order homogeneous equations, namely:

$$\alpha_0(t,x)U + \alpha_1(t,x)\frac{dU}{dx} + \cdots + \frac{d^m U}{dx^m} = 0, \tag{8}$$

where $\alpha_0(t,x), \ldots, \alpha_{m-1}(t,x)$ are arbitrary smooth functions and the variable t is considered as a parameter. It is known that the general solution of (8) has the form

$$U = \varphi_0(t)g_0(t,x) + \cdots + \varphi_{m-1}(t)g_{m-1}(t,x), \tag{9}$$

where $\varphi_0(t), \varphi_1(t), \ldots, \varphi_{m-1}(t)$ are arbitrary functions and $g_0(t,x) = 1$, $g_1(t,x), \ldots$, $g_{m-1}(t,x)$ are fixed functions that form a fundamental system of solutions of (8). Note that in many cases the functions $g_0(t,x)$, ..., $g_{m-1}(t,x)$ can be expressed in an explicit form in terms of elementary ones.

Consider relation (9) as an ansatz for PDEs of the form (5). It is important to note that this ansatz contains m unknown functions $\varphi_i, i = 1, \ldots, m$ that yet-to-be determined. This enables us to reduce a given PDE of the form (5) to a quasilinear system of ODEs of the first order for the unknown functions φ_i. It is well known that such systems have been investigated in detail.

Let us apply ansatz (9) to equation (5). Indeed, calculating with the help of ansatz (9) the derivatives U_t, U_x, U_{xx} and substituting them into PDE (5) , one obtains a very cumbersome expression. But, if one groups similar terms in accordance with powers of the functions $\varphi_i(t)$, then sufficient conditions for reduction of this expression to a system of ODEs can be found. These sufficient conditions have the following form:

$$\lambda g_{i,xx} + s g_i - g_{i,t} = g_{i_1} Q_{ii_1}(t), \tag{10}$$

$$\lambda_0 g_i g_{i,xx} + r(g_{i,x})^2 + p g_i g_{i,x} + q(g_i)^2 = g_{i_1} R_{ii_1}(t), \tag{11}$$

$$\lambda_0(g_i g_{i_1,xx} + g_{i_1} g_{i,xx}) + 2r g_{i,x} g_{i_1,x} + p(g_i g_{i_1,x} + g_{i_1} g_{i,x}) + 2q g_i g_{i_1} = g_j T^j_{ii_1}(t),\ i < i_1, \tag{12}$$

where $Q_{ii_1}, R_{ii_1}, T^j_{ii_1}$ on the right-hand side are defined by the expressions on the left-hand side. The indices t and x of functions $g_i(t,x)$ and $g_{i_1}(t,x)$, $i, i_1 = 0, \ldots, m-1$, denote differentiating with respect to t and x.

With help of conditions (10)–(12), the following system of ODEs is obtained

$$\frac{d\varphi_i}{dt} = Q_{i_1 i}\varphi_{i_1} + R_{i_1 i}(\varphi_{i_1})^2 + T^i_{i_1 i_2}\varphi_{i_1}\varphi_{i_2} + \delta_{i,0}s_0, \quad i = 0, \ldots, m-1 \tag{13}$$

to find the unknown functions $\varphi_i, i = 0, \ldots, m-1$ ($\delta_{i,0} = 0, 1$ is the Kronecker symbol). On the right-hand sides of relations (10)–(12) and (13), a summation is assumed from 0 to m-1 over the repeated indices i_1, i_2, j. So, we have obtained the following statement.

Theorem 1. *Any solution of system (13) generates the exact solution of the form (9) for the nonlinear PDE (5) if the functions $g_i, i = 0, \ldots, m-1$, satisfy conditions (10)–(12).*

Remark 1. The suggested method can be realized for systems of PDEs (see [7, 8], [10]) and for PDEs with derivatives of second and higher orders with respect to t. In the last case, one will obtain systems of ODEs of second and higher orders.

Remark 2. If the coefficients $\lambda_0, \lambda, r, p, q, s$, and s_0 in equation (5) are smooth functions of the variable t, then Theorem 1 is true too. But in this case, the systems of ODEs with time-dependent coefficients are obtained.

3. Construction of the families of non-Lie exact solutions of some nonlinear equations.

Since a *constructive method* for finding new ansätze and exact solutions is suggested, its efficiency will be shown by the examples below. In fact, let us use Theorem 1 for the construction of new exact solutions.

Consider an additional generating condition of third order of the form

$$\alpha_1(t)\frac{dU}{dx} + \alpha_2(t)\frac{d^2U}{dx^2} + \frac{d^3U}{dx^3} = 0, \tag{14}$$

which is the particular case of (8) for $m = 3$. Condition (14) generates the following chain of the ansätze:

$$U = \varphi_0(t) + \varphi_1(t)\exp(\gamma_1(t)x) + \varphi_2(t)\exp(\gamma_2(t)x) \tag{15}$$

if $\gamma_{1,2}(t) = \frac{1}{2}(\pm(\alpha_2^2 - 4\alpha_1)^{1/2} - \alpha_2)$ and $\gamma_1 \neq \gamma_2$;

$$U = \varphi_0(t) + \varphi_1(t)\exp(\gamma(t)x) + x\varphi_2(t)\exp(\gamma(t)x) \tag{16}$$

if $\gamma_1 = \gamma_2 = \gamma \neq 0$;

$$U = \varphi_0(t) + \varphi_1(t)x + \varphi_2(t)\exp(\gamma(t)x) \tag{17}$$

if $\alpha_1 = 0$;

$$U = \varphi_0(t) + \varphi_1(t)x + \varphi_2(t)x^2 \tag{18}$$

if $\alpha_1 = \alpha_2 = 0$.

Remark 3. In the case $D = \alpha_2^2 - 4\alpha_1 < 0$, one obtains complex functions $\gamma_1 = \gamma_2^* = \frac{1}{2}(\pm i(-D)^{1/2} - \alpha_2)$, $i^2 = -1$ and then ansatz (15) is reduced to the form

$$U = \varphi_0(t) + \left[\psi_1(t)\cos\left(\frac{1}{2}(-D)^{1/2}x\right) + \psi_2(t)\sin\left(\frac{1}{2}(-D)^{1/2}x\right)\right]\exp\left(-\frac{\alpha_2 x}{2}\right), \tag{19}$$

where $\varphi_0(t), \psi_1(t), \psi_2(t)$ are yet-to-be determined functions.

Example 1. Consider the following equation

$$U_t = (\lambda + \lambda_0 U)U_{xx} + \lambda_1 U U_x + \lambda_2 U - \lambda_3 U^2 \tag{20}$$

which in the case $\lambda = 1$, $\lambda_0 = 0$ coincides with the Murray equation (4). Hereinafter, it is supposed that $\lambda_2 \neq 0$ since the case $\lambda_2 = 0$ is very especial and was considered in [8]. By substituting the functions $g_0 = 1$, $g_1 = \exp(\gamma_1(t)x)$, $g_2 = \exp(\gamma_2(t)x)$ from ansatz (15) into relations (10)–(12), one can obtain

$$\begin{cases} Q_{00} = \lambda_2, \quad Q_{11} = \lambda\gamma_1^2 + \lambda_2, \quad Q_{22} = \lambda\gamma_2^2 + \lambda_2, \\ R_{00} = -\lambda_3, \quad T_{01}^1 = -\lambda_3, \quad T_{02}^2 = -\lambda_3 \end{cases} \tag{21}$$

and the following relations

$$R_{ii_1} = Q_{ii_1} = T_{ii_1}^j = 0 \tag{22}$$

for different combinations of the indices i, i_1, j. With the help of relations (21)–(22), system (13) is reduced to the form

$$\begin{cases} \dfrac{d\varphi_0}{dt} = \lambda_2\varphi_0 - \lambda_3\varphi_0^2, \\ \dfrac{d\varphi_1}{dt} = (\lambda\gamma_1^2 + \lambda_2)\varphi_1 - \lambda_3\varphi_0\varphi_1, \\ \dfrac{d\varphi_2}{dt} = (\lambda\gamma_2^2 + \lambda_2)\varphi_2 - \lambda_3\varphi_0\varphi_2. \end{cases} \tag{23}$$

The system of ODEs (23) is nonlinear, but it is easily integrated and yields the general solutions

$$\varphi_0 = \frac{\lambda_2}{\lambda_3 + c_0\exp(-\lambda_2 t)}, \quad \varphi_1 = \frac{c_1\exp(\lambda\gamma_1^2 t)}{\lambda_3 + c_0\exp(-\lambda_2 t)}, \quad \varphi_2 = \frac{c_2\exp(\lambda\gamma_2^2 t)}{\lambda_3 + c_0\exp(-\lambda_2 t)}. \quad (24)$$

In (24) and hereinafter, c_0, c_1, c_2 are arbitrary constants . So, by substituting relations (24) into ansatz (15) the three-parameter family of exact solutions of the nonlinear equation (20) for $\lambda_1^2 + 4\lambda_0\lambda_3 \neq 0$ is obtained, namely:

$$U = \frac{\lambda_2 + c_1\exp(\lambda\gamma_1^2 t + \gamma_1 x) + c_2\exp(\lambda\gamma_2^2 t + \gamma_2 x)}{\lambda_3 + c_0\exp(-\lambda_2 t)} \quad (25)$$

where γ_1 and γ_2 are roots of the algebraic equation

$$\lambda_0\gamma^2 + \lambda_1\gamma - \lambda_3 = 0, \qquad \lambda_1^2 + 4\lambda_0\lambda_3 \neq 0. \quad (26)$$

Remark 4. In the case $\lambda_1^2+4\lambda_0\lambda_3 = -\gamma_0^2 < 0$, the complex values γ_1 and γ_2 are obtained, and then the following family of solutions

$$U = \frac{\lambda_2 + \exp\left[\frac{\lambda}{4\lambda_0^2}(\lambda_1^2 - \gamma_0^2)t - \frac{\lambda_1}{2\lambda_0}x)\right](c_1\cos\omega + c_2\sin\omega)}{\lambda_3 + c_0\exp(-\lambda_2 t)}, \quad (27)$$

where $\omega = \dfrac{\gamma_0}{2\lambda_0^2}(\lambda\lambda_1 t - \lambda_0 x)$, are constructed (see ansatz (19)).

Similarly, by substituting the functions $g_0 = 1$, $g_1 = \exp(\gamma(t)x)$, $g_2 = x\exp(\gamma(t)x)$ from ansatz (16) into relations (10)–(12), the corresponding values of the functions R_{ii_1}, Q_{ii_1}, $T^j_{ii_1}$ are obtained, for which system (13) generates the three-parameter family of exact solutions of the nonlinear equation (20) for $\lambda_1^2 + 4\lambda_0\lambda_3 = 0$, namely:

$$U = \frac{\lambda_2 + (c_1 + 2c_2\lambda\gamma t)\exp(\lambda\gamma^2 t + \gamma x) + c_2 x\exp(\lambda\gamma^2 t + \gamma x)}{\lambda_3 + c_0\exp(-\lambda_2 t)}. \quad (28)$$

Analogously, we obtain with the help of ansatz (17) the following family of solutions of equation (20) (at $\lambda_3 = 0$)

$$U = \frac{c_1 + \lambda_2 x + c_2\exp(\lambda\gamma^2 t + \gamma x)}{-\lambda_1 + c_0\exp(-\lambda_2 t)}, \quad (29)$$

where $\gamma = -\dfrac{\lambda_1}{\lambda_0}$.

Finally, ansatz (18) gives the three-parameter family of exact solutions

$$U = \frac{c_2 + 2\lambda\lambda_2 t + c_1 x + \lambda_2 x^2}{-2\lambda_0 + c_0\exp(-\lambda_2 t)} \quad (30)$$

of equation (20) for the case $\lambda_1 = \lambda_3 = 0$, i.e.,

$$U_t = (\lambda + \lambda_0 U)U_{xx} + \lambda_2 U. \quad (31)$$

As is noted, equation (20) for $\lambda = 1$ and $\lambda_0 = 0$ yields the Murray equation (4). If we apply Theorem 1 and ansatz (15) for constructing exact solutions of the Murray equation, then the consraint $\varphi_2 = 0$ is obtained and the two-parameter family of solutions

$$U = \frac{\lambda_2 + c_1 \exp(\gamma^2 t + \gamma x)}{\lambda_3 + c_0 \exp(-\lambda_2 t)}, \tag{32}$$

where $\gamma = \dfrac{\lambda_3}{\lambda_1}$, is found. It is easily seen that the family of exact solutions (25) generates this family if one puts formally $c_2 = 0$, $\lambda = 1$, and $\lambda_0 = 0$ in (25) and (26).

Solutions of the form (32) are not of the plane wave form, but in the case $\lambda_1 < 0$ and $\lambda_3 > 0$, they have similar properties to the plane wave solutions, which were illustrated in [1, 2] in Figures. So, they describe similar processes. In the case $c_0 = 0$, a one-parameter family of plane wave solutions is obtained from (32).

Taking into account solution (32), one obtains the following theorem:

Theorem 2. *The exact solution of the boundary-value problem for the Murray equation (4) with the conditions*

$$U(0, x) = \frac{\lambda_2 + c_1 \exp(\gamma x)}{\lambda_3 + c_0}, \tag{33}$$

$$U(t, 0) = \frac{\lambda_2 + c_1 \exp(\gamma^2 t)}{\lambda_3 + c_0 \exp(-\lambda_2 t)}, \tag{34}$$

and

$$U_x(t, +\infty) = 0, \quad \gamma = \frac{\lambda_3}{\lambda_1} < 0 \tag{35}$$

is given in the domain $(t, x) \in [0, +\infty) \times [0, +\infty)$ *by formula (32), and, for* $\lambda_2 < -\gamma^2$, *it is bounded.*

Note that the Neumann condition (35) (the zero flux on the boundary) is a typical request in the mathematical biology (see, e.g., [1, 2]).

Example 2. Let us consider the following equation

$$U_t = [(\lambda + \lambda_0 U) U_x]_x + \lambda_2 U - \lambda_3 U^2 \tag{36}$$

that, in the case where $\lambda_0 = 0$ and $\lambda = 1$, coincides with the Fisher equation (3). The known soliton-like solution of the Fisher equation was obtained in [11]. Note that this solution can be found using the suggested method too. It turns out that the case $\lambda_0 \neq 0$ is very special.

Let us apply Theorem 1 to construction of exact solutions of equation (36) in the case of ansatz (15). Similarly to Example 1, one can find the following two-parameter families of solutions

$$U = \frac{\lambda_2}{2\lambda_3}\left[1 + \tanh\frac{\lambda_2(t - c_0)}{2}\right] + c_2 \frac{\exp\frac{(2\lambda\lambda_3 + \lambda_0\lambda_2)t}{4\lambda_0}}{\left(\cosh\frac{\lambda_2(t-c_0)}{2}\right)^{3/2}} \exp\left(\sqrt{\frac{\lambda_3}{2\lambda_0}}x\right) \tag{37}$$

and

$$U = \frac{\lambda_2}{2\lambda_3}\left[1 + \tanh\frac{\lambda_2(t - c_0)}{2}\right] + c_1\frac{\exp\frac{(2\lambda\lambda_3+\lambda_0\lambda_2)t}{4\lambda_0}}{\left(\cosh\frac{\lambda_2(t-c_0)}{2}\right)^{3/2}}\exp\left(-\sqrt{\frac{\lambda_3}{2\lambda_0}}x\right), \tag{38}$$

where c_0, c_1, c_2 are arbitrary constants. The solutions from (38) have nice properties. Indeed, any solution U^* of the form (38) holds the conditions $U^* \to \dfrac{\lambda_2}{\lambda_3}$ if $t \to \infty$ and $\lambda\lambda_3 < \lambda_0\lambda_2$; $U^* \to \dfrac{\lambda_2}{2\lambda_3}\left[1 + \tanh\dfrac{\lambda_2(t - c_0)}{2}\right] < 1$ if $x \to +\infty, \lambda_0\lambda_3 > 0$. Taking into account these properties, we obtain the following theorem.

Theorem 3. *The bounded exact solution of the boundary-value problem for the generalized Fisher equation*

$$U_t = [(1 + \lambda_0 U)U_x]_x + \lambda_2 U - \lambda_2 U^2, \qquad \lambda_0 > 1,\ \lambda_2 > 0, \tag{39}$$

with the initial condition

$$U(0, x) = C_0 + C_1\exp\left(-\sqrt{\frac{\lambda_2}{2\lambda_0}}|x|\right), \tag{40}$$

and the Neumann conditions

$$U_x(t, -\infty) = 0, \quad U_x(t, +\infty) = 0, \tag{41}$$

is given in the domain $(t, x) \in [0, +\infty) \times (-\infty, +\infty)$ by the formula

$$U = \frac{1}{2}\left[1 + \tanh\frac{\lambda_2(t - c_0)}{2}\right] + c_1\frac{\exp\frac{\lambda_2(2+\lambda_0)t}{4\lambda_0}}{\left(\cosh\frac{\lambda_2(t-c_0)}{2}\right)^{3/2}}\exp\left(-\sqrt{\frac{\lambda_2}{2\lambda_0}}|x|\right), \tag{42}$$

where $C_0 = \dfrac{1}{2}\left[1 + \tanh\dfrac{-\lambda_2 c_0}{2}\right]$, $C_1 = c_1\left(\cosh\dfrac{-\lambda_2 c_0}{2}\right)^{-3/2}$, *and* $c_1 > 0$.

Example 3. Consider the nonlinear reaction-diffusion equation with a convection term

$$Y_t = [Y^\alpha Y_x]_x + \lambda_1(t)Y^\alpha Y_x + \lambda_2 Y - \lambda_3 Y^{1-\alpha}, \qquad \alpha \neq 0, \tag{43}$$

that can be interpreted as a generalization of the Fisher and Murray equations. One can reduce this equation to the form

$$U_t = UU_{xx} + \frac{1}{\alpha}U_x^2 + \lambda_1(t)UU_x + \alpha\lambda_2 U - \alpha\lambda_3 \tag{44}$$

using the substitution $U = Y^\alpha$. It turns out that equation (44) for $\lambda_1(t) = -\left(1 + \dfrac{1}{\alpha}\right)\gamma(t)$ is reduced by ansatz (17) to the following system of ODEs:

$$\begin{cases} \dfrac{d\gamma}{dt} = -\dfrac{1}{\alpha}\gamma^2\varphi_1, \\ \dfrac{d\varphi_0}{dt} = -\left(1 + \dfrac{1}{\alpha}\right)\gamma\varphi_0\varphi_1 + \alpha\lambda_2\varphi_0 + \dfrac{1}{\alpha}\varphi_1^2 - \alpha\lambda_3, \\ \dfrac{d\varphi_1}{dt} = \alpha\lambda_2\varphi_1 - \left(1 + \dfrac{1}{\alpha}\right)\gamma\varphi_1^2, \\ \dfrac{d\varphi_2}{dt} = \left[-\dfrac{1}{\alpha}\gamma^2\varphi_0 + \left(\dfrac{1}{\alpha} - 1\right)\gamma\varphi_1 + \alpha\lambda_2\right]\varphi_2 \end{cases} \tag{45}$$

for finding the unknown functions $\gamma(t)$ and φ_i, $i = 0, 1, 2$. It is easily seen that in this case the function $\gamma(t) \neq \text{const}$ if $\varphi_1 \neq 0$. Solving the system of ODEs (45), the family of exact solutions is found that are not the ones with separated variables, i.e.,

$$U = \varphi_0(t)g_0(x) + \cdots + \varphi_{m-1}(t)g_{m-1}(x). \tag{46}$$

It is easily seen that in the case $\alpha = -1$, the system of ODEs (45) is integrated in terms of elementary functions and one finds $\gamma(t) = [\lambda_2 c_0 + c_1 \exp(-\lambda_2 t)]^{-1}$.

Remark 5. The family of exact solutions with $\gamma(t) \neq \text{const}$ (see ansatz (17)) has an essential difference from the ones obtained above since they contain the function $\gamma(t)$. So this family cannot be obtained using the method of linear invariant subspaces recently proposed in [12, 13] (note that the basic ideas of the method used in [12, 13] were suggested in [14]) because that method is reduced to finding solutions in the form (46).

Finally, it is necessary to observe that all found solutions are not of the form (7), i.e., they are not plane wave solutions. Moreover, all these solutions except (30) can not be obtained using the Lie method. One can prove this statement using theorems that have been obtained in [15].

4. Discussion

In this paper, a constructive method for obtaining exact solutions of certain classes of nonlinear equations arising in mathematical biology was applied. The method is based on the consideration of a fixed nonlinear partial differential equation together with *additional generating condition* in the form of a linear high-order ODE. With the help of this method, new exact solutions were obtained for nonlinear equations of the form (20), which are generalizations of the Fisher and Murray equations.

As follows from Theorems 2 and 3, the found solutions can be used for construction of exact solutions of some boundary-value problems with zero flux on the boundaries. Similarly to Theorem 3, it is possible to obtain theorems for construction of periodic solutions and blow-up solutions of some boundary-value problems with zero flux on the boundaries.

The efficiency of the suggested method can be shown also by construction of exact solutions to nonlinear reaction-diffusion systems of partial differential equations. For example, it is possible to find ones for the well-known systems of the form (see, e.g., [16])

$$\begin{cases} \lambda_1 U_t = \Delta U + U\dfrac{f_1(U,V)}{f_2(U,V)}, \\ \lambda_2 V_t = \Delta V + V\dfrac{g_1(U,V)}{g_2(U,V)}, \end{cases} \tag{47}$$

where the f_k and g_k, $k = 1, 2$, are linear functions of U and V. The form of the found solutions will be essentially depend on coefficients in the functions f_k and g_k, $k = 1, 2$. Note that, in the particular case, system (47) gives the nonlinear system

$$\begin{cases} \lambda_1 U_t = \Delta U + \beta_1 U^2 V^{-1}, \\ \lambda_2 V_t = \Delta V + \beta_2 U, \quad \beta_1 \neq \beta_2. \end{cases} \tag{48}$$

As was shown in [17, 18], system (48) has the wide Lie symmetry. Indeed, it is invariant with respect to the same transformations as *the linear heat equation*. This fact gives additional wide possibilities for finding families of exact solutions.

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