Letter to the Editor

# Another integrable case in the Lorenz model

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**Abstract**

A scaling invariance in the Lorenz model allows one to consider the usually discarded case $\sigma = 0$. We integrate it with the third Painlevé function.

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## 1 Introduction

The Lorenz model [1]

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sigma(y-x),\ \frac{\mathrm{d}y}{\mathrm{d}t} = rx - y - xz,\ \frac{\mathrm{d}z}{\mathrm{d}t} = xy - bz, \tag{1}$$

in which $(b, \sigma, r)$ are real constants, is a prototype of chaotic behaviour [3]. In particular, it fails the Painlevé test unless the parameters obey the constraints [4]

$$Q_2 \equiv (b-2\sigma)(b+3\sigma-1) = 0, \tag{2}$$

$$\begin{aligned}\forall x_2 : Q_4 \equiv\ & -4i(b-\sigma-1)(b-6\sigma+2)x_2 - (4/3)(b-3\sigma+5)b\sigma r \\ & +(-4+10b+30b^2-20b^3-16b^4)/27 \\ & +(-38b-56b^2-(28/3)b^3+88\sigma+86b^2\sigma)\sigma/3 \\ & -32\sigma/9+70b\sigma^2-64\sigma^3-58b\sigma^3+36\sigma^4 = 0.\end{aligned} \tag{3}$$

This system (2)–(3) depends on $r$ only through the product $b\sigma r$, as a consequence of an obvious scaling invariance in the model, and it admits four solutions,

$$(b, \sigma, b\sigma r) = (1, 1/2, 0), (2, 1, 2/9), (1, 1/3, 0), (1, 0, 0). \tag{4}$$

In the first three cases, i.e. when the system (1) is nonlinear, which excludes $\sigma = 0$, the system can be explicitly integrated [4], and the general solution $(x, y, z)$ is a singlevalued function of time expressed with, respectively, an elliptic function, the second and the third Painlevé functions.

In this letter, we consider the fourth case

$$(b, \sigma, r) = (1, 0, r). \tag{5}$$

The apparently linear nature of the dynamical system can be removed by eliminating $y$ and $z$ and considering the third order differential equation for $x(t)$ [5],

$$y = x + x'/\sigma,\ z = r - 1 - [(\sigma+1)x' + x'']/(\sigma x), \tag{6}$$

$$\begin{aligned}& xx''' - x'x'' + x^3x' + \sigma x^4 + (b+\sigma+1)xx'' + (\sigma+1)(bxx' - x'^2) \\ & +b(1-r)\sigma x^2 = 0,\end{aligned} \tag{7}$$

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which also depends on $r$ only through the product $b\sigma r$, and thus implements the above mentioned scaling invariance. The necessary conditions for (7) to pass the Painlevé test are the same ($Q_2 = 0, Q_4 = 0$) as for the dynamical system (1), the restriction $\sigma \neq 0$ being now removed.

## 2 Integration for $b = 1, \sigma = 0$

Because of the scaling invariance, the following first integral [4] of the dynamical system (1),

$$(b, \sigma, r) = (1, \sigma, 0): \ K_3 = (y^2 + z^2)e^{2t}, \tag{8}$$

is also a first integral of the third order equation for $(b, \sigma, b\sigma r) = (1, \sigma, 0)$, which includes the particular case of interest to us $(b, \sigma, b\sigma r) = (1, 0, 0)$,

$$(b, \sigma, b\sigma r) = (1, 0, 0): \ K^2 = \lim_{\sigma \to 0} \sigma^2 K_3 = \left[ \left( \frac{x'' + x'}{x} \right)^2 + x'^2 \right] e^{2t}. \tag{9}$$

For $K = 0$, the general solution is

$$x = ik \tanh \frac{k}{2}(t - t_0) - i, \ i^2 = -1, \ (k, t_0) \text{ arbitrary.} \tag{10}$$

For $K \neq 0$, after taking the usual parametric representation

$$\frac{x'' + x'}{x} = Ke^{-t} \cos \lambda, \ x' = Ke^{-t} \sin \lambda, \tag{11}$$

the second order ODE for $\lambda(t)$ is found to be

$$\lambda'' - Ke^{-t} \sin \lambda = 0, \tag{12}$$

with the link

$$x(t) = \lambda'(t). \tag{13}$$

In the variable $\cos \lambda$, the differential equation (12) becomes algebraic and belongs to an already integrated class [2]. The overall result is the general solution

$$x = i + 2i \frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Log} w(\xi(t)), \ i^2 = -1, \ \xi = ae^{-t}, \tag{14}$$

in which $w(\xi)$ is the particular third Painlevé function defined by

$$\frac{\mathrm{d}^2 w}{\mathrm{d}\xi^2} = \frac{1}{w} \left( \frac{\mathrm{d}w}{\mathrm{d}\xi} \right)^2 - \frac{\mathrm{d}w}{\xi \mathrm{d}\xi} + \frac{\alpha w^2 + \gamma w^3}{4\xi^2} + \frac{\beta}{4\xi} + \frac{\delta}{4w}, \tag{15}$$

$$\alpha = 0, \ \beta = 0, \ \gamma\delta = -(K/a)^2. \tag{16}$$

## 3 Conclusion

Out of the two cases selected by the condition $Q_2 = 0$, one admits a first integral [4],

$$b = 2\sigma: \ K_1 = (x^2 - 2\sigma z)e^{2\sigma t}, \tag{17}$$

but, in the second case $b = 1 - 3\sigma$, the first integral whose existence has been conjectured [6] is not yet known. The present result, which belongs to this unsettled case $b = 1 - 3\sigma$, should help to solve this open question.

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