Letter to the Editor

Another integrable case in the Lorenz model

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Abstract

A scaling invariance in the Lorenz model allows one to consider the usually discarded case
\( \sigma = 0 \). We integrate it with the third Painlevé function.

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1 Introduction

The Lorenz model \[1\]

\[
\frac{dx}{dt} = \sigma(y-x), \quad \frac{dy}{dt} = rx - y - xz, \quad \frac{dz}{dt} = xy - bz,
\]

in which \((b, \sigma, r)\) are real constants, is a prototype of chaotic behaviour \[3\]. In particular, it fails
the Painlevé test unless the parameters obey the constraints \[4\]

\[
Q_2 \equiv (b - 2\sigma)(b + 3\sigma - 1) = 0,
\]

\( \forall x_2 : Q_4 \equiv -4i(b - \sigma - 1)(b - 6\sigma + 2)x_2 - (4/3)(b - 3\sigma + 5)b\sigma r \\
+(-4 + 10b + 30b^2 - 20b^3 - 16b^4)/27 \\
+(-38b - 56b^2 - (28/3)b^3 + 88b + 86b^2\sigma)\sigma/3 \\
-32\sigma/9 + 70b\sigma^2 - 64\sigma^3 - 58b\sigma^3 + 36\sigma^4 = 0.
\]

This system \[2\]–\[3\] depends on \( r \) only through the product \( b\sigma r \), as a consequence of an obvious
scaling invariance in the model, and it admits four solutions,

\[ (b, \sigma, b\sigma r) = (1, 1/2, 0), (2, 1, 2/9), (1, 1/3, 0), (1, 0, 0). \]

In the first three cases, i.e. when the system \[1\] is nonlinear, which excludes \( \sigma = 0 \), the system
can be explicitly integrated \[4\], and the general solution \((x, y, z)\) is a singlevalued function of time
expressed with, respectively, an elliptic function, the second and the third Painlevé functions.

In this letter, we consider the fourth case

\[ (b, \sigma, r) = (1, 0, r). \]

The apparently linear nature of the dynamical system can be removed by eliminating \( y \) and \( z \) and
considering the third order differential equation for \( x(t) \) \[4\],

\[
y = x + x'/\sigma, \quad z = r - 1 - [(\sigma + 1)x' + x'']//(\sigma x),
\]

\[
x x''' - x' x'' + x^3 x' + 3 x^2 + (b + \sigma + 1) x x' + (\sigma + 1)(b x x' - x^2) \\
+b(1-r)\sigma x^2 = 0,
\]

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which also depends on \( r \) only through the product \( b\sigma r \), and thus implements the above mentioned scaling invariance. The necessary conditions for (7) to pass the Painlevé test are the same \((Q_2 = 0, Q_4 = 0)\) as for the dynamical system (1), the restriction \( \sigma \neq 0 \) being now removed.

2 Integration for \( b = 1, \sigma = 0 \)

Because of the scaling invariance, the following first integral (4) of the dynamical system (1),

\[
(b, \sigma, r) = (1, \sigma, 0) : \quad K_3 = (y^2 + z^2)e^{2t},
\]

is also a first integral of the third order equation for \((b, \sigma, b\sigma r) = (1, \sigma, 0)\), which includes the particular case of interest to us \((b, \sigma, b\sigma r) = (1, 0, 0)\),

\[
(b, \sigma, b\sigma r) = (1, 0, 0) : \quad K^2 = \lim_{\sigma \to 0} \sigma^2 K_3 = \left[ \left( \frac{x'' + x'}{x} \right)^2 + x'^2 \right] e^{2t}.
\]

For \( K = 0 \), the general solution is

\[
x = ik \tanh \frac{k}{2} (t - t_0) - i, \quad i^2 = -1, \quad (k, t_0) \text{ arbitrary}.
\]

For \( K \neq 0 \), after taking the usual parametric representation

\[
\frac{x'' + x'}{x} = Ke^{-t} \cos \lambda, \quad x' = Ke^{-t} \sin \lambda,
\]

the second order ODE for \( \lambda(t) \) is found to be

\[
\lambda'' - Ke^{-t} \sin \lambda = 0,
\]

with the link

\[
x(t) = \lambda'(t).
\]

In the variable \( \cos \lambda \), the differential equation (12) becomes algebraic and belongs to an already integrated class (2). The overall result is the general solution

\[
x = i + 2i \frac{d}{dt} \log w(\xi(t)), \quad i^2 = -1, \quad \xi = ae^{-t},
\]

in which \( w(\xi) \) is the particular third Painlevé function defined by

\[
\frac{d^2 w}{d\xi^2} = \frac{1}{w} \left( \frac{dw}{d\xi} \right)^2 - \frac{dw}{\xi d\xi} + \frac{\alpha w^2 + \gamma w^3 + \beta}{4\xi^2} + \frac{\delta}{4w},
\]

\[
\alpha = 0, \quad \beta = 0, \quad \gamma \delta = -(K/a)^2.
\]

3 Conclusion

Out of the two cases selected by the condition \( Q_2 = 0 \), one admits a first integral (4),

\[
b = 2\sigma : \quad K_1 = (x^2 - 2\sigma z)e^{2\sigma t},
\]

but, in the second case \( b = 1 - 3\sigma \), the first integral whose existence has been conjectured (3) is not yet known. The present result, which belongs to this unsettled case \( b = 1 - 3\sigma \), should help to solve this open question.

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References


