

Letter to the Editor

Another integrable case in the Lorenz model

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Abstract

A scaling invariance in the Lorenz model allows one to consider the usually discarded case $\sigma = 0$. We integrate it with the third Painlevé function.

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1 Introduction

The Lorenz model [1]

$$\frac{dx}{dt} = \sigma(y - x), \quad \frac{dy}{dt} = rx - y - xz, \quad \frac{dz}{dt} = xy - bz, \quad (1)$$

in which (b, σ, r) are real constants, is a prototype of chaotic behaviour [3]. In particular, it fails the Painlevé test unless the parameters obey the constraints [4]

$$\begin{aligned} Q_2 &\equiv (b - 2\sigma)(b + 3\sigma - 1) = 0, & (2) \\ \forall x_2 : Q_4 &\equiv -4i(b - \sigma - 1)(b - 6\sigma + 2)x_2 - (4/3)(b - 3\sigma + 5)b\sigma r \\ &\quad + (-4 + 10b + 30b^2 - 20b^3 - 16b^4)/27 \\ &\quad + (-38b - 56b^2 - (28/3)b^3 + 88\sigma + 86b^2\sigma)\sigma/3 \\ &\quad - 32\sigma/9 + 70b\sigma^2 - 64\sigma^3 - 58b\sigma^3 + 36\sigma^4 = 0. & (3) \end{aligned}$$

This system (2)–(3) depends on r only through the product $b\sigma r$, as a consequence of an obvious scaling invariance in the model, and it admits four solutions,

$$(b, \sigma, b\sigma r) = (1, 1/2, 0), (2, 1, 2/9), (1, 1/3, 0), (1, 0, 0). \quad (4)$$

In the first three cases, i.e. when the system (1) is nonlinear, which excludes $\sigma = 0$, the system can be explicitly integrated [4], and the general solution (x, y, z) is a singlevalued function of time expressed with, respectively, an elliptic function, the second and the third Painlevé functions.

In this letter, we consider the fourth case

$$(b, \sigma, r) = (1, 0, r). \quad (5)$$

The apparently linear nature of the dynamical system can be removed by eliminating y and z and considering the third order differential equation for $x(t)$ [5],

$$y = x + x'/\sigma, \quad z = r - 1 - [(\sigma + 1)x' + x'']/(\sigma x), \quad (6)$$

$$\begin{aligned} &xx''' - x'x'' + x^3x' + \sigma x^4 + (b + \sigma + 1)xx'' + (\sigma + 1)(bxx' - x'^2) \\ &+ b(1 - r)\sigma x^2 = 0, \end{aligned} \quad (7)$$

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which also depends on r only through the product $b\sigma r$, and thus implements the above mentioned scaling invariance. The necessary conditions for (7) to pass the Painlevé test are the same ($Q_2 = 0, Q_4 = 0$) as for the dynamical system (1), the restriction $\sigma \neq 0$ being now removed.

2 Integration for $b = 1, \sigma = 0$

Because of the scaling invariance, the following first integral [4] of the dynamical system (1),

$$(b, \sigma, r) = (1, \sigma, 0) : K_3 = (y^2 + z^2)e^{2t}, \quad (8)$$

is also a first integral of the third order equation for $(b, \sigma, b\sigma r) = (1, \sigma, 0)$, which includes the particular case of interest to us $(b, \sigma, b\sigma r) = (1, 0, 0)$,

$$(b, \sigma, b\sigma r) = (1, 0, 0) : K^2 = \lim_{\sigma \rightarrow 0} \sigma^2 K_3 = \left[\left(\frac{x'' + x'}{x} \right)^2 + x'^2 \right] e^{2t}. \quad (9)$$

For $K = 0$, the general solution is

$$x = ik \tanh \frac{k}{2}(t - t_0) - i, \quad i^2 = -1, \quad (k, t_0) \text{ arbitrary.} \quad (10)$$

For $K \neq 0$, after taking the usual parametric representation

$$\frac{x'' + x'}{x} = Ke^{-t} \cos \lambda, \quad x' = Ke^{-t} \sin \lambda, \quad (11)$$

the second order ODE for $\lambda(t)$ is found to be

$$\lambda'' - Ke^{-t} \sin \lambda = 0, \quad (12)$$

with the link

$$x(t) = \lambda'(t). \quad (13)$$

In the variable $\cos \lambda$, the differential equation (12) becomes algebraic and belongs to an already integrated class [2]. The overall result is the general solution

$$x = i + 2i \frac{d}{dt} \text{Log } w(\xi(t)), \quad i^2 = -1, \quad \xi = ae^{-t}, \quad (14)$$

in which $w(\xi)$ is the particular third Painlevé function defined by

$$\frac{d^2 w}{d\xi^2} = \frac{1}{w} \left(\frac{dw}{d\xi} \right)^2 - \frac{dw}{\xi d\xi} + \frac{\alpha w^2 + \gamma w^3}{4\xi^2} + \frac{\beta}{4\xi} + \frac{\delta}{4w}, \quad (15)$$

$$\alpha = 0, \quad \beta = 0, \quad \gamma\delta = -(K/a)^2. \quad (16)$$

3 Conclusion

Out of the two cases selected by the condition $Q_2 = 0$, one admits a first integral [4],

$$b = 2\sigma : K_1 = (x^2 - 2\sigma z)e^{2\sigma t}, \quad (17)$$

but, in the second case $b = 1 - 3\sigma$, the first integral whose existence has been conjectured [6] is not yet known. The present result, which belongs to this unsettled case $b = 1 - 3\sigma$, should help to solve this open question.

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