Differential constraints and exact solutions of nonlinear diffusion equations

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Abstract

The differential constraints are applied to obtain explicit solutions of nonlinear diffusion equations. Certain linear determining equations with parameters are used to find such differential constraints. They generalize the determining equations used in the search for classical Lie symmetries.

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1. Introduction.

Differential constraints arisen originally in the theory of partial differential equations of the first order. In particular Jacobi used differential constraints to find the total integral of nonlinear equation

$$F(x_1, ..., x_n, z, z_{x_1}, ..., z_{x_n}) = 0,$$

König applied them to the equation of the second order [1]. They required that the corresponding over-determined system was compatible. The general theory of over-determined systems was developed by Delassus, Riquier, Cartan, Ritt, Kuranishi, Spencer and others. One can find references in the book of Pommaret [2]. Now the applications of over-determined systems include such diverse fields as differential geometry, continuum mechanics and nonlinear optics.

The general formulation of the method of differential constraints requires that the original system of partial differential equations

$$F^1 = 0, ..., \quad F^m = 0 \tag{1}$$

be enlarged by appending additional differential equations (differential constraints)

$$h_1 = 0, ..., \quad h_p = 0, \tag{2}$$

such that the over-determined system (1), (2) satisfies some conditions of compatibility.

One can derive many exact solutions of partial differential equation by means of differential constraints. It was particularly shown in [3] that some soliton solutions can be found using differential constraints. Olver and Rosenau [4], Olver [5], Kaptsov [3], Levi and Winternitz [6] show that many reduction methods such as non-classical symmetry groups, partial invariance, separation of variables, the Clarkson–Kruskal direct method can be included into the method of differential constraints. In practice, methods based on the Riquier–Ritt theory of over-determined systems of partial differential equations may be difficult. The problem of finding all differential constraints compatible with certain equations can be more complicated than the investigation of the original equations.

It was recently proposed a new method for finding differential constraints which uses linear determining equations. These equations are more general than the classical determining equations for Lie generators [7] and depend on some parameters. Given an evolution equation

$$u_t = F(t, x, u, u_1, \dots, u_n), \tag{3}$$

where $u_k = \frac{\partial^k u}{\partial x^k}$, then according to [8] the linear determining equation corresponding to (3) is of the form

$$D_t(h) = \sum_{i=0}^{N} \sum_{k=0}^{i} b_{ik} D_x^{i-k}(F_{u_{N-k}}) D_x^{N-i}(h), b_{ik} \in R. \tag{4}$$

Here and throughout D_t, D_x are the operators of total differentiation with respect to t and x. Equality (4) must hold for all solutions of (3). The function h may depends on $t, x, u, u_1, \dots, u_p$. The number p is called the order of the solution of equation (4). If we have some solution h then corresponding differential constraint is

$$h = 0. \tag{5}$$

It was also shown in [8] that equations (4) and (5) constitute the compatible system. Thus we sketch the derivation of some solutions to the evolution equation (4).

(I) Find solutions of the linear determining equations (4).

(II) Fixing the function h, we obtain differential constrain (5).

(III) Find the general solution of (5) which includes some arbitrary functions a_i depending on t.

(IV)Substitute the general solution into (4). It leads to ordinary differential equations for functions a_i.
(V) Solve the ordinary differential equations and obtain a solution of the evolution equation (4).

In this paper we start with determination of the solutions of linear determining equations of the second and third orders for the nonlinear diffusion equation

$$u_t = (u^k u_x)_x + f(u). \tag{6}$$

These solutions exist only if f belongs to the special forms. Then we use the obtained functions h to find solutions of equation (6). In final section we derive exact solutions of two-dimension equation

$$u_t = \Delta \ln(u).$$

2. Solutions of linear determining equations.

The nonlinear diffusion equation

$$u_t = (Q(u)u_x)_x + f(u). \tag{7}$$

often arises in the description of various physical processes. The group classification of the equation has been carried out in [9]. Some exact solutions of (7) can be found in [10, 11]. In physical applications Q is usually taken to be a power function. In this section we consider the equation

$$u_t = (u^q u_x)_x + f(u), \tag{8}$$

where f is an differentiable function, $q \neq 0$. If $q = -2$, $f = u$ or $f = const$ then the equation (8) can be linearized. We shall not discuss this case here.

The linear determining equation, which corresponds to (8), is

$$D_t h = u^q D_x^2 h + b_1 q u_x u^{q-1} D_x h + (b_3 q u^{q-1} u_{xx} + b_2 q(q-1)u^{q-2}u_x^2 + b_4 f_u)h, \tag{9}$$

where $b_1, \ldots, b_4 \in R$. We shall seek solutions to (9) in the form

$$h = u_n + g(t, x, u, \ldots, u_{n-1}),$$

where $n \geq 2$, $u_k = \frac{\partial^k u}{\partial x^k}$. The method for finding solutions is very similar to the standard procedure applied in the group analysis of differential equations [12] and only one of all possibilities is described here for the sake of brevity.

We set $n = 2$. First, let us express all t-derivatives in (9) using (8). As a result, the left-hand side of (3.4) becomes a polynomial with respect to u_3, u_2. The polynomial must identically vanish. Collecting similar terms we obtain the following relations for the coefficients of u_3 and u_2^2

$$q(b_1 - 4) = 0, \quad u g_{u_1 u_1} + q(b_3 - 3) = 0.$$

Thus $b_1 = 4$ and g can be represented as follows

$$g = \frac{(3 - b_3)q}{2u} u_1^2 + a(u, t, x)u_1 + g_1(u, t, x);$$

here a and g_1 must be functions of u, t and x alone. Collecting the coefficients of $u_2 u_1^2$ and $u_2 u_1$, we have the equations

$$2b_2 q - 2b_2 - b_3^2 q + b_3 q + 4b_3 - 6q = 0, \tag{10}$$

$$2u a_u + q(b_3 + 1)a = 0.$$

In follows from the last equation that

$$a = a_1(t, x) u^{-\frac{(1+b_3)}{2} q},$$

where a_1 is a function of t and x. Next we consider the coefficient u_1^3 and obtain equation

$$4b_2 q - 4b_2 + b_3^2 q - 4b_3 q + 2b_3 - 9q + 6 = 0. \tag{11}$$

From (10) and (11) it follows that $b_3 = 1$ or $b_3 = \frac{q+2}{q}$.

Assuming $b_3 = 1$, we obtain $b_2 = \frac{3q-2}{q-1}$. The coefficient of u_2 give us equation

$$u^{q+2}(2a_{1_x} + f_u(b_4 - 1)) + u^{2q+1} q g_1 = 0$$

The equation enables us to express

$$g_1 = \frac{1}{q} u^{1-q}(f_u(1 - b_4) - 2a_{1_x})$$

The coefficient of u_1^2 yields Euler equation

$$u^3(1 - b_4) f_{uuu} + u^2(2 - qb_4 - 2b_4) f_{uu} - uq^2 f_u + q^2 f = 0.$$

Consider for simplicity the case $b_4 = 1$. It is easy to see that the last equation has two types of solutions:

$$f = ku + nu^{-q}, \qquad q \neq -1$$

or

$$f = ku + nu \ln u, \qquad q = -1$$

where k, n are arbitrary constants. Let us focus on $f = ku + nu^{-q}$. It follows from above calculations and equation (9) that

$$u^{q+1}(-qa_{1_t} - 3u^q qa_{1_{xx}} - 4u^q a_{1_{xx}} + kq^2 a_1)u_1 +$$
$$+nq^2 a_1 u_1 + 2u^{q+2}(a_{1_{tx}} - u^q a_{1_{xxx}} - kqa_{1_x}) + 2una_{1_x} = 0. \qquad (12)$$

From (12) we have $na_1 = 0$.

If $a_1 = 0$ then the solution of (9) is $h = u_2 + qu_1^2/u$. If $n = 0$ and $q \neq -4/3$ then one easily computes

$$a_1 = (rx + s)\exp(kqt),$$

$$h = u_2 + q\frac{u_1^2}{u} + \left((rx + s)u_1 - \frac{2}{q}ur\right)u^{-q}\exp(kqt).$$

In the case $q = -4/3$, we obtain

$$a_1 = (rx^2 + sx + p)\exp(-\frac{4}{3}kt),$$

$$h = u_2 - \frac{4u_1^2}{3u} + \left((rx^2 + sx + p)u_1 + \frac{3}{2}u(2rx + s)\right)u^{4/3}\exp(-\frac{4}{3}kt).$$

It can be shown that the found functions h lead to invariant solutions of the corresponding equation (8).

We omit here for the sake of brevity intermediate calculations and give the list of solutions to the equation (9):

(1) if $q = -1$ and $f = su + ru\ln(u)$ then

$$h = u_2 - \frac{u_1^2}{u};$$

(2) if $q \neq -1$ and $f = su + ru^{-q}$ then

$$h = u_2 + \frac{qu_1^2}{u};$$

(3) if $q = -2$ and $f = su + ru^3$ then

$$h = u_2 - \frac{3u_1^2}{2u};$$

(4) if $q=1$ and $f=ru$ then

$$h=u_2+s\exp(rt)u^{-2}u_1+r/3;$$

(5) if q is an arbitrary constant and $f=su+ru^{1-q}$ then

$$h=u_2-\frac{(q-1)u_1^2}{u},$$

with $r,s\in R$.
We did not include functions h that correspond to invariant solutions of the equation (8).

If we will look for solutions to the equation (9), which depend on third derivative, then obtain the following list:
(1) if q is an arbitrary constant and $f=su+ru^{1-q}+\frac{n(q+1)}{q^2}u^{q+1}$ then

$$h=u_3+\frac{3(q-1)}{u}u_1u_2+(q^2-3q+2)\frac{u_1^3}{u^2}+nu_1;$$

(2a) if $q\neq 1$ and $f=nu+\frac{r}{q}u^{q+1}$ then

$$h=u_3+\frac{(3q-1)}{u}u_1u_2+q(q-2)\frac{u_1^3}{u^2}+ru_1; \qquad (13)$$

(2b) if $q=-2$ or $q=-4/3$ and $f=nu+\frac{r}{q}u^{q+1}+mu^{q+3}$ then h is also given by (13);
(3) if $q=-\frac{1}{2}$ and $f=mu$ then

$$h=u_3-\frac{5u_1u_2}{2u}+\frac{5u_1^3}{4u^2}+r\exp(-3mt/2)u^{5/2}+s\exp(mt/2)u^{1/2};$$

(4) if $q=-\frac{3}{2}$ and $f=nu+mu^{5/2}$ then

$$h=u_3-\frac{15u_1u_2}{2u}+\frac{35u_1^3}{4u^2}+r\exp(-3nt/2)u^{5/2};$$

(5) if $q=-\frac{1}{2}$ and $f=mu-2ku^{1/2}$ then

$$h=u_3-\frac{5u_1u_2}{2u}+\frac{5u_1^3}{4u^2}+ku_1+s\exp(mt/2)u^{1/2};$$

(6) if $q=-\frac{3}{2}$ and $f=nu$ then

$$h=u_3-\frac{15u_1u_2}{2u}+\frac{35u_1^3}{4u^2}+s\exp(-7n/2t)u^{9/2}+r\exp(-3nt/2)u^{5/2};$$

(7) if $q=-1$ and $f=mu$ then

$$h = u_3 - \frac{4u_1u_2}{u} + \frac{3u_1^3}{u^2} + s\exp(-2mt)u^2u_1;$$

with $r, s, m, n \in R$. Here we also did not include functions h leading to invariant solutions of (8).

3. Solutions of diffusion equations.

In this section we shall use the functions obtained above to construct solutions of diffusion equations (8). One can apply the method described in introduction.

We first take the function $h = u_2 + qu_1^2/u$, where $q \in R$, corresponding to some cases mentioned above. Simply by equating this function to zero, we obtain the differential constraint

$$u_2 + qu_1^2/u = 0. \tag{14}$$

The equation (14) has two types of solutions:

$$u = (c_1x + c_2)^{\frac{1}{q+1}}, \qquad q \neq -1, \tag{15}$$

$$u = c_1\exp(c_2x), \qquad q = -1, \tag{16}$$

where c_1, c_2 are functions of t.

If we substitute the representation (16) into equation

$$u_t = (u_x/u)_x + ku\ln u \tag{17}$$

then this leads us to differential equations for c_1, c_2. From this equations it is easy to find c_1, c_2 and obtain the following solution of (17)

$$u = s_1\exp(s_2x\exp(kt)), \qquad s_1, s_2 \in R.$$

Substituting (15) into equation

$$u_t = (u^qu_x)_x + su + ru^{-q},$$

we find the solution

$$u = \exp(st)\Big(ax + b - \frac{r}{s(q+1)}\exp(-s(q+1)t)\Big)^{\frac{1}{q+1}},$$

with $a, b \in R$.

It is easy to see that the differential constraint

$$u_2 + (q-1)\frac{u_1^2}{u} = 0$$

for equation

$$u_t = (u^q u_x)_x + ru + su^{1-q}$$

leads to solution

$$u = \left(qax\exp(qrt) + \frac{a^2}{r}\exp(2qrt) - \frac{s}{r}\right)^{\frac{1}{q}}, \quad a \in R.$$

Now let us consider the differential constraints of the third order. We start with the equation

$$u_t = (u^q u_x)_x + su + ru^{1-q} + n\frac{q+1}{q^2}u^{q+1}, \quad n \in R \tag{18}$$

As explained above this equation is compatible with the differential constraint

$$u_3 + 3(q-1)\frac{u_1 u_2}{u} + (q^2 - 3q + 2)\frac{u_1^3}{u^2} + nu_1 = 0. \tag{19}$$

By a change of variable $v = u^q$ one may rewrite (18), (19) in the following way

$$v_t = vv_{xx} + \frac{1}{q}v_x^2 + n\frac{q+1}{q}v^2 + sqv + rq, \tag{20}$$

$$v_3 + nv_1 = 0. \tag{21}$$

If $n = -1$ then it follows from (21) that

$$v(t, x) = a + b\exp(x) + c\exp(-x), \tag{22}$$

where a, b and c are some functions of t. Substituting this representation into equation (20) we obtain the system of ordinary differential equations for the function a, b and c:

$$a_t = -a^2(1 + 1/q) + asq - 4bc/q + rq, \tag{23}$$

$$b_t = -ab(1 + 2/q) + bsq, \tag{24}$$

$$c_t = -ac(1 + 2/q) + csq. \tag{25}$$

Using finite-dimensional invariant subspaces, Galaktionov [13] found representation (22).

From (24) and (25) we derive the first integral $b = kc,\ k \in R$. Therefore the system (23)-(25) can be reduced to nonlinear ordinary differential equation for the function a. In general we can not express solutions of (23)-(25) in terms of the elementary functions. We give one example of exact solution of equation (20), with $q = -1$ and $r = 0$. This solution has the representation (22) and the functions a, b, c are

$$a = a_1 \sin(pr_1 \exp(r_1 t) + m) \exp(r_1 t) / \cos(pr_1 \exp(r_1 t) + m),$$

$$b = a_2 \exp(r_1 t) / \cos(pr_1 \exp(r_1 t) + m),$$

$$c = a_3 \exp(r_1 t) / \cos(pr_1 \exp(r_1 t) + m),$$

where $a_1 = pr_1^2$, $a_3 = p^2 r_1^4 / 4a_2$, $r_1 = -s$ and p, a_2, m are arbitrary constants.

Now let us consider the equation

$$u_t = (u^{-1/2} u_x)_x + mu - 2k\sqrt{u}, \qquad m, k \in R \tag{26}$$

and the differential constraint

$$u_3 - \frac{5u_1 u_2}{2u} + \frac{5u_1^3}{4u^2} + ku_1 + se^{mt/2}\sqrt{u} = 0. \tag{27}$$

Using the equation (26), one can write (27) as

$$(\ln u)_{tx} + se^{mt/2} u^{-1} = 0.$$

Replacing $\ln(e^{mt/2} u^{-1})$ by w, the last equation is replaced by the Liouville equation

$$w_{tx} = s \exp(w).$$

Since the general solution of the Liouville equation is

$$w = \ln \frac{2T'X'}{s(T+X)^2},$$

it gives the representation

$$u = \frac{s(T+X)^2}{2T'X'} e^{mt/2}, \tag{28}$$

where T and X are the arbitrary functions of t and x respectively. Substituting this representation into (26), we have

$$\sqrt{\frac{s}{2}}\, e^{mt/4}\Big(2(T')^{1/2}-(T+X)(T')^{-3/2}T''-\frac{m}{2}(T+X)(T')^{-1/2}\Big)=$$

$$=\frac{3}{2}(X')^{-3/2}(X'')^2-(X')^{-1/2}X'''-2k(X')^{1/2}. \qquad (29)$$

Differentiating (29) with respect to t, it is easy to obtain the equation for T

$$2T'''T'-3(T'')^2+\frac{m^2}{4}(T')^2=0.$$

If $m \neq 0$ then the function

$$T=c_1\tanh\Big(\frac{mt}{4}+c_3\Big)+c_2, \quad c_1,c_2,c_3\in R$$

is the general solution of this equation. Substituting the function T into (29), we get the following equation for X

$$\sqrt{\frac{sm}{2c_1}}\Big(c_1-c_2-X\Big)=\frac{3}{2}(X')^{-3/2}(X'')^2-(X')^{-1/2}X'''-2k(X')^{1/2}.$$

It should be noted that (28) is equivalent to the representation

$$u=(a_1+a_2e^{mt/2})^2,$$

where a_1, a_2 are functions of x. This representation yields the following system for a_1 and a_2

$$a_{1_{xx}}+a_1^2m/2-a_1k=0, \qquad (30)$$

$$a_{2_{xx}}+a_1a_2m/2-a_2k=0. \qquad (31)$$

In general, it is possible to express a_1 in terms of the Weierstrass function $\wp$ and a_2 in terms of Lamé's function [14] . However, if $m = 12$ and $k = 4$, then the functions

$$a_1=\frac{1}{\cosh^2(x)},$$

$$a_2=\frac{a}{\cosh^2(x)}+\frac{b}{\cosh^2(x)}\Big(\frac{\sinh 4x}{32}+\frac{\sinh 2x}{2}+\frac{3x}{8}\Big), \qquad a,b\in R$$

satisfy the equations (30), (31).

According to our results in the previous section, as $k=0$, the equation (26) is compatible with the differential constraint

$$u_3 - \frac{5u_1u_2}{2u} + \frac{5u_1^3}{4u^2} + re^{-3mt/2}u^{5/2} + se^{mt/2}\sqrt{u} = 0. \qquad (32)$$

Using the equation (26), one can write (32) as

$$(\ln u)_{tx} + re^{-3mt/2}u + se^{mt/2}u^{-1} = 0. \qquad (33)$$

Replacing $\ln(e^{-3mt/2}u)$ by w in (33) yields

$$w_{tx} + e^w + sre^{-w-mt} = 0.$$

If we set $s=0$ then from the last equation we find the following representation

$$u = -\frac{2X'T'}{(X+T)^2}e^{3mt/2},$$

where T and X are the arbitrary functions of t and x respectively. Substituting this representation into (26) leads to equation

$$\sqrt{-2/r}e^{3mt/4}\Big(-m(T')^{1/2}(X+T) - 2(T')^{-1/2}T''(X+T) + 4(T')^{3/2}\Big) =$$

$$= -2X'''(X')^{-3/2}(X+T)^2 + 8X''(X')^{-1/2}(X+T) - 8(X')^{3/2} + (X'')^2(X')^{-5/2}(X+T)^2$$

Introducing new functions

$$C(T) = \sqrt{-2/r}e^{3mt/4}\Big(m(T')^{1/2} + 2T''(T')^{-1/2}\Big),$$

$$B(X) = (X'')^2(X')^{-5/2} - 2X'''(X')^{-3/2},$$

$$D(X) = 2XB + 8X''(X')^{-1/2},$$

$$Q(T) = \sqrt{-2/r}e^{3mt/4}\Big(m(T')^{1/2}T + 2(T')^{-1/2}T''T - 4(T')^{3/2}\Big),$$

$$R(X) = BX^2 + 8X''(X')^{-1/2}X - -8(X')^{3/2},$$

one can write the last equation as

$$C(T)X + D(X)T + B(X)T^2 + Q(T) + R(X) = 0. \qquad (34)$$

It is possible consider (34) as condition of orthogonality of two vector functions $Z=(C,T,T^2,Q,1), \quad W=(X,D,B,1,R)$.

Denote by $\rho(Z)$ and $\rho(W)$ the number of linearity independent functions among $C, T, T^2, Q, 1$ and $X, D, B, R, 1$ respectively. From orthogonality condition it follows that $\rho(Z) + \rho(W) \leq 5$. It is possible to show that if $T' \neq 0$ then $\rho(Z) = 3$ and $\rho(W) = 2$. In this case we have

$$D(X) = a_1 X + b_1, \quad B(X) = a_2 X + b_2, \quad R(X) = a_3 X + b_3,$$

with $a_i, b_i \in R$. Because of (34) and definition of the functions D, B, R we obtain equations

$$(X')^3 = (c_3 X^3 + c_2 X^2 + c_1 X + c_0)^2, \tag{35}$$

$$(T')^3 = A(-c_3 T^3 + c_2 T^2 - c_1 X + c_0)^2, \tag{36}$$

where c_3, c_2, c_1 and c_0 are arbitrary constants, $A = (-2r)^{1/3}$.

The solutions of (35) and (36) can be expressed in the terms of the Weierstrass function $\wp$ [15]. Indeed, one can write (35) and (36) as

$$(X')^3 = (c_3(X - \alpha_1)(X - \alpha_2)(X - \alpha_3))^2, \tag{38}$$

$$(T')^3 = A(-c_3(T + \alpha_1)(T + \alpha_2)(T + \alpha_3))^2.$$

Replacing $X - \alpha_1$ by $1/Y$ in (38) yields

$$(Y')^3 + B^2 (Y - b_1)^2 (Y - b_2)^2 = 0,$$

where $B = c_3(\alpha_2 - \alpha_1)(\alpha_3 - \alpha_1)$, $b_1 = \frac{1}{\alpha_2 - \alpha_1}$, $b_2 = \frac{1}{\alpha_3 - \alpha_1}$. Introducing new function Z such that $Z^3 = B(Y - b_1)(Y - b_2)$, we obtain equation

$$(Z')^2 + \frac{4B}{9} Z^3 + \frac{B^2 (b_1 - b_2)^2}{9} = 0. \tag{39}$$

The solutions of the last equation are expressed in the terms of the Weierstrass function $\wp$. Applying the above process to (36), we obtain equation such as (39).

We shall omit here other cases and discuss briefly in the next section two-dimensional equation.

4. Two-dimensional equation.

We consider here the fast diffusion equation

$$u_t = \Delta \ln(u). \tag{40}$$

Some applications of this equation can be found in [16]. Galaktionov [10] used invariant subspaces to find some solutions of (40). If we set $u = 1/v$, we obtain

$$v_t = v^2 \Delta \ln(v). \tag{41}$$

It is easy to check that one of elementary solutions of (41) is travelling wave given by

$$v = 1 + c \exp(mx + ny - (m^2 + n^2)t), \tag{42}$$

where c, m and n are arbitrary constants. Obviously, this is invariant solution. On the other hand, this solution satisfies differential constraints

$$v_x - mv + m = 0,$$

$$v_y - nv + n = 0.$$

It is possible to find other differential constraints that are linear with respect x, y and v.

It can be shown that the differential constraints

$$v_x + \frac{c_1 - c_0 tan(t)}{c_0^2 + c_1^2}\left(v - xc_0 - yc_1 + t(c_0^2 + c_1^2)\right) = c_0,$$

$$v_y - \frac{c_0 + c_1 tan(t)}{c_0^2 + c_1^2}\left(v - xc_0 - yc_1 + t(c_0^2 + c_1^2)\right) = c_1$$

are compatible with the equation (41). Here c_0 and c_1 are arbitrary constants. The solution of (41) corresponding to these constraints is

$$v = c_0 x + c_1 y - (c_0^2 + c_1^2)t + c_2 \cos(t) \exp(m_1 x + m_2 y + m_3 t).$$

Here c_0, c_1, c_2 are arbitrary constants and

$$m_1 = \frac{c_0 t - c_1}{c_0^2 + c_1^2}, \quad m_2 = \frac{c_1 t + c_0}{c_0^2 + c_1^2}, \quad m_3 = -tan(t).$$

We can derive other explicit solutions using invariant subspaces [10] or linear differential constraints. For example, from [10] one may extract the following representation

$$v = s_0 + s_1 \cos(x) + s_2 \sin(x) + s_3 \exp(y) + s_4 exp(-y),$$

where functions $s_i(t)$ satisfy ordinary differential equations

$$s_0' + s_1^2 + s_2^2 - 4s_3 s_4 = 0, \tag{43}$$

$$s_1' + s_1 s_0 = 0,$$
$$s_2' + s_2 s_0 = 0, \tag{44}$$
$$s_3' - s_3 s_0 = 0, \tag{45}$$
$$s_4' - s_4 s_0 = 0.$$

Because of (44) and (45) we find

$$s_3'/s_2' + s_3/s_2 = 0.$$

This yields

$$s_2 = c_2/s_3, \quad c_2 \in R.$$

By arguments similar to that used above we have

$$s_1 = c_1/s_3,\ s_4 = c_4 s_3, \quad c_1, c_4 \in R.$$

Substituting this into (43) leads to

$$s_0' + (c_1^2 + c_2^2)s_3^{-2} - 4c_4 s_3^2 = 0.$$

From (45) we express the function s_0 and obtain

$$(\ln s_3)'' = a s_3^2 - b s_3^{-2}, \tag{46}$$

with $a = 4c_4,\ b = c_1^2 + c_2^2$.

Setting $a = b = 1$ one can derive two elementary solutions

$$s_3 = tanh(t), \qquad s_3 = tan(t)$$

In general, the solutions of (46) can be expressed in terms of elliptic functions. It is easy to obtain the correspondent function u.

The more complicated representation is

$$v = s_0 + s_1\cos(2x) + s_2\sin(2x) + s_3\exp(2y) + s_4\exp(-2y) +$$

$$s_5\sin(x)\exp(y) + s_6\sin(x)\exp(-y) + s_7\cos(x)\exp(y) + s_8\cos(x)\exp(-y),$$

where s_i are functions which satisfy some ordinary differential equations. The special case of this representation was found in [10].

It is important to note that the equation (40) is invariant under infinite-dimensional algebra of symmetry [9]. Some solutions of (40) were obtained by means of these symmetries in [9]. We shall describe other method of using symmetry. It is convenient to apply the complex conjugate variables $z = x + iy$, $\bar{z} = x - iy$. Thus, we can write the equation (40) as

$$u_t = \frac{1}{4}\frac{\partial^2 u}{\partial z \partial \bar{z}}. \tag{47}$$

It is easy to check that (47) is invariant under the transformation

$$z' = A(z), \quad \bar{z}' = B(\bar{z}), \quad u' = u/(A_z B_{\bar{z}}),$$

where $A(z)$ and $B(\bar{z})$ are arbitrary functions. In other words, if the function $f(t, z, \bar{z})$ is a solution of (47) then $f(t, A(z), B(\bar{z}))A_z B_{\bar{z}}$ also satisfies (47).

For example, if we set $m = n = 1$ then from (42) we can construct the solution of the equation (40)

$$u = \frac{A_z \bar{A}_{\bar{z}}}{1 + c\exp(A + \bar{A} - 2t)},$$

where A is an arbitrary function of z and $\bar{A}$ is the complex conjugate function.

5. Conclusions.

In sections 2 and 3 we have shown how the method of the linear determining equations can be applied to find explicit solutions to nonlinear diffusion equations. We have found exact solutions of these equations, using only the simplest solutions of the linear determining equations. It is interesting to find solutions of the linear determining equations depending on derivatives of higher orders. A.Shmidt [17, 18] applied this method to another parabolic equations and some systems; application to the elliptic equation is discussed in [8].

In section 4 we have considered the two-dimensional equation. Applications of systems of the linear determining equations to multi-dimensional equations briefly discussed in [19]. Using results of section 3 one can find the following representation

$$u = (a + be^{mt/2})^2. \tag{48}$$

of solution of the equation

$$u_t = \Delta(u^{1/2}) + mu + nu^{1/2}, \qquad m, n \in R,$$

where the functions $a(x, y)$ and $b(x, y)$ must satisfy the system

$$\Delta a = ma^2 + na,$$

$$\Delta b = mab + nb.$$

It is easy to show that the differential constraint

$$u_{tt} = u_t^2/2u + mu_t/2$$

leads to the representation (48). The interesting reductions of some diffusion equations in several independent variables can be found in [20, 21]. It is important to explain these reductions by means of differential constraints.

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