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Generalised Symmetries and the Ermakov-Lewis Invariant

R GOODALL † *and P G L LEACH* ‡1

† *School of Mathematical and Statistical Sciences, Howard College, University of KwaZulu-Natal, Durban 4041, Republic of South Africa*

‡ *Department of Mathematics, Luleå University of Technology, Luleå SE-971 87, Sweden*
E-mail: leachp@ukzn.ac.za

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Abstract

Generalised symmetries and point symmetries coincide for systems of first-order ordinary differential equations and are infinite in number. Systems of linear first-order ordinary differential equations possess a generalised rescaling symmetry. For the system of first-order ordinary differential equations corresponding to the time-dependent linear oscillator the invariant of this symmetry has the form of the famous Ermakov-Lewis invariant, but in fact reveals a richer structure.

The origins of the linear second-order ordinary differential equation known as the time-dependent linear oscillator are disparately manifold. A classical source is the lengthening pendulum described in the normal approximation by

$$\ddot{\theta} + \omega^2(t)\theta = 0. \tag{0.1}$$

(The pendulum has to be one of increasing length. Otherwise the approximation $\sin\theta \approx \theta$ breaks down [36, 35].) At the first Solvay Conference in 1911 Lorentz proposed an adiabatic invariant for (0.1) based on its Hamiltonian representation as

$$I_{adiabatic} = \frac{1}{2\omega(t)}\left(\dot{\theta}^2 + \omega^2(t)\theta^2\right) \tag{0.2}$$

in the case that $\omega^2(t)$ was a slowly varying function. The precise mathematical qualities of this adiabatic invariant were delineated some half-century later by the English mathematician Littlewood [25, 26, 27, 28].

In the search for workable confinement devices for controlled thermonuclear fusion the time-dependent linear oscillator again made its appearance as the model equation for the motion of a charged particle in an axially symmetric electromagnetic field. In 1966 Lewis [21, 22] applied Kruskal's asymptotic method [14] to find an invariant which would be an improvement on the adiabatic invariant in that it would apply to a wider class

[1]permanent address: School of Mathematical and Statistical Sciences, Howard College, University of KwaZulu-Natal, Durban 4041, Republic of South Africa

of functions $\omega^2(t)$ than the slowly varying function of the adiabatic invariant. To his considerable surprise the zeroth-order term was the only nonzero term in the asymptotic expansion. The Lewis invariant, as it was promptly termed [3], for (0.1) was

$$I = \tfrac{1}{2}\left[\left(\rho\dot{\theta} - \dot{\rho}\theta\right)^2 + \left(\frac{\theta}{\rho}\right)^2\right], \tag{0.3}$$

where $\rho(t)$ was any solution to the second-order nonlinear ordinary differential equation

$$\ddot{\rho} + \omega^2(t)\rho = \frac{1}{\rho^3} \tag{0.4}$$

which has become known as the Pinney equation after Pinney presented its solution as

$$\rho^2 = Au^2 + 2Buv + Cv^2, \quad AC - B^2 = \frac{1}{W^2}, \tag{0.5}$$

where u and v were two independent solutions of (0.1) and W is their Wronskian [34]. In fact (0.5) is intimately involved with the theory of linear third-order ordinary differential equations of maximal symmetry [29] since (0.4) comes from the integration of

$$\dddot{y} + \omega^2\dot{y} + \omega\dot{\omega}y = 0, \quad y = \tfrac{1}{2}\rho^2 \tag{0.6}$$

when the constant of integration is set at 2. The general solution of (0.6) is (0.5) without the constraint on the constants A, B and C [30].

The Hamiltonian corresponding to (0.1) is

$$H = \tfrac{1}{2}\left(p_\theta^2 + \omega^2(t)\theta^2\right), \quad p_\theta = \dot{\theta}. \tag{0.7}$$

The invariant (0.3) was demonstrated by Leach [15] simply to be the expression in (θ, t) variables of the Hamiltonian

$$\tilde{H} = \tfrac{1}{2}\left(P_\Theta^2 + \Theta^2\right) \tag{0.8}$$

which, being autonomous, is automatically a first integral and to which (0.7) is related by the Generalised Canonical Transformation [2]

$$\Theta = \frac{\theta}{\rho}, \quad P_\Theta = \rho\dot{\theta} - \dot{\rho}\theta, \quad T = \int \rho^{-2}\mathrm{d}t. \tag{0.9}$$

The use of the Generalised Canonical Transformation was extended to potentials other than quadratic by Lewis and Leach [23] and González-Gascón *et al* [5] and to dimension greater than one by Grammaticos *et al* [8] and Lewis [24].

All of this time, approximately 100 years, the invariant (0.3) had already been demonstrated by the Ukrainian mathematician V Ermakov [4] who in 1880 considered the two equations

$$\ddot{x} + \omega^2(t)x = 0 \tag{0.10}$$

$$\ddot{y} + \omega^2(t)y = \frac{1}{y^3} \tag{0.11}$$

and, on eliminating the $\omega^2(t)$ by multiplying (0.10) by y and x by (0.11), subtracting the two and then multiplying by the integrating factor $\dot{x}y - x\dot{y}$, arrived at the first integral

$$I = \tfrac{1}{2}\left[(\dot{x}y - x\dot{y})^2 + \frac{1}{y^2}\right]. \tag{0.12}$$

It was not until the late seventies of the last century that the work of Ermakov became widely known.

One should note that (0.1) is the normal form of a scalar linear second-order ordinary differential equation and so its solution has relevance in a wide area of Mathematical Physics for which the model equation is, or reduces to, a linear second-order ordinary differential equation.

In both the theory and practice of ordinary differential equations it is a commonplace to reduce an higher-order equation (or system) to a system of first-order ordinary differential equations. This is the classical approach to proving theorems on the existence of solutions [11] [pp 72-73] and the not so classical way to determine complete symmetry groups [12] for systems of higher-order ordinary differential equations with an insufficient number of point symmetries for the purpose [31, 32]. The transformation from an higher-order equation to a system of first-order ordinary differential equations is not a point transformation and so there is no preservation of the point symmetry properties of the original equation. In fact for a first-order ordinary differential equation a point symmetry is equally a generalised symmetry and there exists an infinite number for any given first-order ordinary differential equation or system of first-order ordinary differential equations. In the particular case of a scalar linear nth-order ordinary differential equation the natural reduction to a system of first-order ordinary differential equations produces a system of linear first-order ordinary differential equations. (In the case of nonlinear nth-order ordinary differential equations which are linearisable to linear nth-order ordinary differential equations the natural reduction may not be so obvious without a knowledge of the linearising transformation. One notes that the linearising transformation need not be a point transformation. a classic example to the contrary is the equation $y'' + 3yy' + y^3 = 0$ which is linearised to $w''' = 0$ by means of the nonlocal transformation $(x, y) \implies (x, w\ :\ y = w'/w.)$

In the case of a linear second-order ordinary differential equation there is already the equivalence class of all linear second-order ordinary differential equations [6]. This is not the case with higher-order equations under point transformations. For example linear third-order ordinary differential equations can have four, five or seven Lie point symmetries. However, when the ‘equivalent’ system of linear first-order ordinary differential equations is considered, there always exists a point transformation (in the variables of the system of first-order ordinary differential equations) to transform one to another (provided that the number of variables is the same!) [32]. The alternate route is to introduce nonlocal transformations to the third- (*æq* higher-) order level to demonstrate the equivalence of the three classes of linear equations when the classification is made in terms of point symmetries [7, 18]. The very fact that first-order ordinary differential equations have an infinite number of Lie point symmetries (equally generalised symmetries for first-order equations) has a negative impact on the value of the existence of a symmetry. Apart from this consideration the general problem of the determination of symmetries of first-order ordinary differential equations is not solvable. However, there are some symmetries which can be considered. One simply imposes a constraint on the form of the symmetry. In the case of systems of linear first-order ordinary differential equations a generalised self-similar symmetry is an obvious symmetry to be considered since there is by definition similarity in the dependent variables and so one has only to determine how the independent variable comes into the symmetry. For examples of a different type of constraint see [1, 9, 10].

Consider the system of homogeneous equations

$$\dot{\mathbf{x}} = A\mathbf{x} \tag{0.13}$$

in which the elements of the coefficient matrix A may be time-dependent. In general (0.13) possesses a Lie point symmetry

$$\Gamma = \tau\partial_t + \eta\partial_{\mathbf{x}} \tag{0.14}$$

if

$$\dot{\eta} = (\tau A)^{\cdot}\mathbf{x} + A\eta. \tag{0.15}$$

Without any specification of the variable dependency in τ and η (0.15) has an infinite number of solutions. However, if we demand that Γ have the form of a generalised self-similar symmetry, *ie* be of the form

$$\Gamma = s_0\partial_t + s_{ij}x_j\partial_{x_i} \tag{0.16}$$

(0.15) becomes

$$\dot{S} = (s_0A)^{\cdot} + [A, S]\,, \tag{0.17}$$

where $S = [s_{ij}]$ and $[A, S]$ is the usual commutator of the matrices A and S.

In contrast to (0.15), (0.17) is reasonably well-defined. There is one redundancy in that there are n^2 first-order equations for the $n^2 + 1$ functions s_0 and s_{ij}. One can consider the differing results of taking, say, $s_0 = 0$ or one of the $s_{ij} = 0$. In the case of linear second-order ordinary differential equations being reduced to a system of two first-order equations it may come as a surprise that the choice $s_0 = 0/s_{12} = 0$ is equivalent and the latter choice is a point symmetry of the original second-order equation. For systems derived from equations of order higher than two this equivalence falls away. At the third-order the choice $s_0 = 0/s_{13} = 0$ relates point symmetries at the first-order level and contact symmetries at the third-order level. At the order of four or more even this is no longer available and one realises that there should be no surprise in variation in the symmetry properties of ordinary differential equations as one goes from second- to third- to higher-order equations.

We now consider the equation of the time-dependent harmonic oscillator, (0.1). As a system of first-order equations this is simply written as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\omega^2 x_1, \end{aligned} \tag{0.18}$$

where $x_1 = \theta$ and $x_2 = \dot{\theta}$. We are now in a position to look at the symmetries of the system (0.18). In general a symmetry of the form (0.14) is a solution of (0.15). For η independent of the x_i, τ must be 0 and one obtains the solution symmetries

$$\Gamma_s = f_i(t)\partial_{x_i}, \tag{0.19}$$

where

$$\dot{f}_i = A_{ij} f_j. \tag{0.20}$$

The solution symmetries are very useful since they enable one to transform from one linear system to another and provide the basis for the unity of all linear (and linearisable) nth-order ordinary differential equations [32].

The solution symmetries (0.19) require the solution of the original system. This is rarely the easiest to accomplish and, varying with the applications, not always precisely relevant. In the case of the generalised similarity symmetry (0.17) with (0.18) becomes

$$\begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}^{\cdot} = \left(s_0 \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \right)^{\cdot} + \left[\begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}, \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \right] \tag{0.21}$$

which is a system of four equations with five variables. If we put $s_0 = 0$, *ie* the transformation of variables induced by the symmetry (0.16) does not change the time, we have the system of equations

$$\begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}^{\cdot} = \begin{bmatrix} B & I \\ -\omega^2 I & B \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}, \tag{0.22}$$

where we have written

$$\mathbf{u} = \begin{bmatrix} s_{11} \\ s_{12} \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} s_{21} \\ s_{22} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & \omega^2 \\ -1 & 0 \end{bmatrix} \tag{0.23}$$

and I is the 2×2 unit matrix. From the system of equations

$$\dot{\mathbf{u}} = B\mathbf{u} + \mathbf{v} \tag{0.24}$$

$$\dot{\mathbf{v}} = -\omega^2 \mathbf{u} + B\mathbf{v} \tag{0.25}$$

we can use (0.24) to eliminate $\mathbf{v}$ from (0.25) and obtain the second-order equation for $\mathbf{u}$, *videlicet*

$$\ddot{\mathbf{u}} - 2B\dot{\mathbf{u}} - \dot{B}\mathbf{u} = 0, \tag{0.26}$$

after a certain amount of simplification. In terms of the components of $\mathbf{u}$ (0.26) becomes

$$\ddot{s}_{11} - 2\omega^2 \dot{s}_{12} - 2\omega\dot{\omega} s_{12} = 0 \tag{0.27}$$

$$\ddot{s}_{12} + 2\dot{s}_{11} = 0 \tag{0.28}$$

and we can decouple the variables to obtain the single third-order equation for s_{12}, *videlicet*

$$\tfrac{1}{2}\dddot{s}_{12} + 2\omega^2 \dot{s}_{12} + 2\omega\dot{\omega} s_{12} = 0 \tag{0.29}$$

which is easily integrated once by means of the integrating factor s_{12} to

$$\tfrac{1}{2}\ddot{s}_{12} s_{12} - \tfrac{1}{4}\dot{s}_{12}^2 + \omega^2 s_{12}^2 = K, \tag{0.30}$$

where K is the constant of integration. The appearance of (0.30) is improved by the introduction of an auxiliary function defined by $s_{12} = \rho^2$. Then we have

$$\ddot{\rho} + \omega^2 \rho = \frac{K}{\rho^3} \tag{0.31}$$

$$s_{11} = -\rho\dot{\rho} \qquad s_{12} = \rho^2 \tag{0.32}$$

$$s_{21} = -\dot{\rho}^2 - \frac{K}{\rho^2} s_{22} = \rho\dot{\rho}, \tag{0.33}$$

where in (0.33) we have used (0.24). We have the generalised self-similar symmetry

$$\Gamma_{ss} = \left(-\rho\dot{\rho}x_1 + \rho^2 x_2\right)\partial_{x_1} + \left(-\left(\dot{\rho}^2 + \frac{K}{\rho^2}\right)x_1 + \rho\dot{\rho}x_2\right)\partial_{x_2}. \tag{0.34}$$

This is not the most general form of the self-similar symmetry since we set $s_0 = 0$, but the additional generality is spurious. It is, as it were, that we have fixed a gauge function. Given Γ_{ss}, it is an easy matter to calculate the invariant of the system of first-order ordinary differential equations (0.18) associated with this symmetry. The invariants of Γ_{ss} are the solutions of the associated Lagrange's system

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x_1}{-\rho\dot{\rho}x_1 + \rho^2 x_2} = \frac{\mathrm{d}x_2}{-\left(\dot{\rho}^2 + \frac{K}{\rho^2}\right)x_1 + \rho\dot{\rho}x_2} \tag{0.35}$$

and are

$$u = t \qquad v = \tfrac{1}{2}\left[\left(\rho x_2 - \dot{\rho}x_1\right)^2 + \frac{K}{\rho^2}\right]. \tag{0.36}$$

In the second of these invariants of the symmetry we recognise the invariant for the system of differential equations, (0.18), as the Ermakov-Lewis invariant. (The presence of the constant K in (0.36) is a consequence of its presence in (0.30).) Naturally, if we did not recognise this, we would have to proceed to the second part of the determination of the invariant by solving the equation

$$\frac{\mathrm{d}v}{\mathrm{d}u} = f(u, v) \tag{0.37}$$

obtained by the differentiation of the two invariants in (0.35). In this case we would find that $f = 0$.

In (0.36) there is in fact some detail which appears to have been generally overlooked. It is conventional to write the Ermakov-Lewis invariant in terms of the auxiliary function $\rho(t)$ which is the solution of the nonlinear second-order ordinary differential equation (0.31) (usually with the constant of integration, K, fixed at 1). However, in our symmetry-based approach the coefficient functions, s_{ij}, are found from the solution of the linear third-order ordinary differential equation (0.29). Consequently there are three linearly independent sets of coefficient functions. That these are given in terms of the two linearly independent solutions of (0.1) is a consequence of (0.29) being a third-order ordinary differential equation of maximal point symmetry and consequently belonging to that hierarchy of ordinary differential equations for which the solutions can be expressed in terms of simple functional combinations of the solutions of (0.1) [30].

The realisation of the three Ermakov-Lewis invariants is obscured by the time dependence of $\omega^2(t)$. For the purposes of our discussion we obtain a clearer picture if we take $\omega^2(t) = 1$. However, we emphasise that there is no loss of generality in the subsequent discussion, simply a gain in clarity. Equation (0.29) is now

$$\dddot{s}_{12} + 4\dot{s}_{12} = 0 \tag{0.38}$$

with the solution set $\{1, \sin 2t, \cos 2t\}$ or, as would be more convenient in quantum mechanical applications [20], $\{1, \exp[2it], \exp[-2it]\}$. There are three sets of coefficient functions, s_{ij}, given by

$$\begin{aligned} s_{11} &= 0 & s_{12} &= 1 \\ s_{21} &= -1 & s_{22} &= 0 \end{aligned} \tag{0.39}$$

corresponding to the solution 1,

$$\begin{aligned} s_{11} &= -\cos 2t & s_{12} &= \sin 2t \\ s_{21} &= \sin 2t & s_{22} &= \cos 2t \end{aligned} \tag{0.40}$$

corresponding to the solution $\sin 2t$ and

$$\begin{aligned} s_{11} &= \sin 2t & s_{12} &= \cos 2t \\ s_{21} &= \cos 2t & s_{22} &= -\sin 2t \end{aligned} \tag{0.41}$$

corresponding to the solution $\cos 2t$, where we have used (0.28) and (0.24) to obtain s_{11}, s_{21} and s_{22} from s_{12}. We note that, although (0.39) follows easily from (0.31–0.33), this is not the case for (0.40) and (0.41) since different combinations of the basis solutions than those chosen here are required. The problem is obviated if the exponential basis set is used.

From these three symmetries three invariants follow. The generalised self-similar symmetries are

$$\begin{aligned} \Gamma_{ss1} &= x_2\partial_{x_1} - x_1\partial_{x_2} \\ \Gamma_{ss2} &= (-x_1\cos 2t + x_2\sin 2t)\partial_{x_1} + (x_1\sin 2t + x_2\cos 2t)\partial_{x_2} \\ \Gamma_{ss3} &= (x_1\sin 2t + x_2\cos 2t)\partial_{x_1} + (x_1\cos 2t - x_2\sin 2t)\partial_{x_2} \end{aligned} \tag{0.42}$$

and for each of the symmetries the corresponding invariant is easily calculated to be

$$\begin{aligned} I_1 &= \tfrac{1}{2}\left(x_1^2 + x_2^2\right) \\ I_2 &= \tfrac{1}{2}\left(x_1^2 - x_2^2\right)\sin 2t + x_1x_2\cos 2t \\ I_3 &= \tfrac{1}{2}\left(x_1^2 - x_2^2\right)\cos 2t - x_1x_2\sin 2t. \end{aligned} \tag{0.43}$$

The integral, I_1, is the usual Ermakov-Lewis invariant of the time-dependent harmonic oscillator when this particular solution is replaced by the solution corresponding to $\omega^2(t)$. The transformation properties of the integral, I_1, and the invariants, I_2 and I_3, in the general case of an n-dimensional oscillator system have received detailed treatment in the past [16].

We note that Γ_{ss1} has the form of a rotation symmetry and this recalls the angular momentum interpretation of the traditional Ermakov-Lewis invariant, which is the invariant of Γ_{ss1}, proposed by Eliezer and Gray in 1976 [3]. Such an interpretation is not obvious for Γ_{ss2} and/or for Γ_{ss3}. Even if we take the generally useful combinations

$$\begin{aligned}\Gamma_{ss\pm} &= \pm i\Gamma_{ss2} + \Gamma_{ss3} \\ &= \mathrm{e}^{\pm 2it}\left[(\pm ix_1 + x_2)\partial_{x_1} + (x_1 \pm ix_2)\partial_{x_2}\right],\end{aligned} \tag{0.44}$$

it is apparently rather hard to force the interpretation of some sort of rotational invariance implied in I_2 and I_3. Nevertheless this is the case since the integrals, (0.43b,c), and the symmetries, (0.44), represent invariant hyperbolic rotations [19], an invariance which is reflected in the transformation properties of I_1, I_2 and I_3 in the context of canonical transformations of Hamiltonian Mechanics [16].

In fine we find that in the representation of the second-order ordinary differential equation for the time-dependent harmonic oscillator as a system of linear first-order ordinary differential equations there are three generalised self-similar symmetries when one imposes the constraint that there be no transformation of the time variable. Not surprisingly the algebra of the three generalised self-similar symmetries is $sl(2,R)$. The same algebra is found for the three invariants in (0.43) under the operation of taking the Poisson Bracket. Although the standard approaches to the Ermakov-Lewis invariant give only one such invariant, a group theoretical approach based on the Lie point symmetries of the corresponding linear system naturally reveals three invariants. In the case of a linear system it makes sense to use the generalised self-similar symmetries as a starting point for an investigation. It is an open question whether a similar approach can work with systems of first-order equations derived from equations of higher order which possess some suitable symmetry property such as self-similarity. Certainly there is some positive evidence that this could be the case, for example in the instance of the Kepler Problem and related problems [31, 33].

Appendix

In the body of the text we used the particular case of $\omega^2(t) = 1$ to provide a simpler and clearer discussion of the explicit forms of the invariants, their algebraic properties and geometrical interpretation. At that time we noted that there was no loss of generality in so doing. Indeed, since the central purpose of this paper is the discussion of the use of the symmetries of systems of first-order equations to obtain invariants, the precise functional form of $\omega^2(t)$ is relevant only if it provides an obstacle to further progress. However, for the benefit of the reader who wishes to see the general results we provide them in this brief Appendix.

The crux of the problem is to find the three linearly independent solutions of (0.29), *videlicet*

$$\dddot{s}_{12} + 4\omega^2\dot{s}_{12} + 4\omega\dot{\omega}s_{12} = 0, \tag{0.45}$$

since the remaining elements of the matrix, B, follow more or less directly from s_{12}. Since (0.45) is a third-order differential equation of maximal point symmetry, three linearly independent solutions are obtained from three linearly independent quadratic combinations of two linearly independent solutions of the second-order differential equation

$$\ddot{u} + \omega^2(t)u = 0. \tag{0.46}$$

Suppose that these two linearly independent solutions of (0.46) are $u(t)$ and $v(t)$. They may be expressed in terms of a solution of the Ermakov-Pinney equation

$$\ddot{\rho} + \omega^2(t)\rho = \frac{1}{\rho^3} \tag{0.47}$$

as

$$u = \rho \sin T \quad \text{and} \quad v = \rho \cos T, \tag{0.48}$$

where

$$T = \int \rho^{-2}(t)\mathrm{d}t. \tag{0.49}$$

To maintain a parallel with the main text we take the three linearly independent solutions for s_{12} to be

$$\sigma_1 = u^2 + v^2 = \rho^2 \tag{0.50}$$

$$\sigma_2 = 2uv = \rho^2 \sin 2T \tag{0.51}$$

$$\sigma_3 = -u^2 + v^2 = \rho^2 \cos 2T. \tag{0.52}$$

The other elements of B follow as in (0.32) and (0.33) with the exception that we find

$$s_{21} = -\left(\dot{\rho}^2 - \frac{1}{\rho^2}\right)\sin 2T - 2\frac{\dot{\rho}}{\rho}\cos 2T \tag{0.53}$$

$$s_{21} = -\left(\dot{\rho}^2 - \frac{1}{\rho^2}\right)\cos 2T + 2\frac{\dot{\rho}}{\rho}\sin 2T \tag{0.54}$$

for (0.51) and (0.52) respectively.

The invariants corresponding to those given in (0.43) are then

$$\begin{aligned} I_1 &= \tfrac{1}{2}\left[(\dot{\rho}x_1 - \rho x_2)^2 + \left(\frac{x_1}{\rho}\right)^2\right] \\ I_2 &= \tfrac{1}{2}\left[(\dot{\rho}x_1 - \rho x_2)^2 - \left(\frac{x_1}{\rho}\right)^2\right]\sin 2T + \frac{x_1}{\rho}(\dot{\rho}x_1 - \rho x_2)\cos 2T \\ I_3 &= \tfrac{1}{2}\left[(\dot{\rho}x_1 - \rho x_2)^2 - \left(\frac{x_1}{\rho}\right)^2\right]\cos 2T - \frac{x_1}{\rho}(\dot{\rho}x_1 - \rho x_2)\sin 2t. \end{aligned} \tag{0.55}$$

The first invariant is the usual Ermakov-Lewis invariant.

The geometric interpretation is somewhat more difficult to depict for general $\omega^2(t)$. However, the algebra of both the three symmetries (Lie Brackets) and the three invariants (Poisson Brackets) remains as $sl(2,R)$ which is better written as its $so(2,1)$ variant for the geometric interpretation of $SO(2,1)$ is that of rotations on an hyperboloid. A closed-form solution of the Ermakov-Pinney equation (0.47) for general $\omega^2(t)$ is not possible. Nevertheless the use of $\rho(t)$ as the source of the required solutions of (0.29) enables one to take most computations to the second last line without the need for extensive numerical calculations as was demonstrated, for example, in a discussion of Berry's Phase for time-dependent Hamiltonian systems [17].

One notes that in the approach adopted here the time remains an invariant of each of the three generalised symmetries used to construct the invariants.

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