NEW SIMILARITY SOLUTIONS OF THE UNSTEADY INCOMPRESSIBLE BOUNDARY-LAYER EQUATIONS

by DAVID K. LUDLOW[†]

(*Flow Control & Prediction Group, College of Aeronautics, Cranfield University, Cranfield, Bedfordshire* MK43 0AL)

PETER A. CLARKSON[‡]

(*Institute of Mathematics and Statistics, University of Kent at Canterbury, Canterbury, Kent* CT2 7NF)

and ANDREW P. BASSOM[§]

(*School of Mathematical Sciences, University of Exeter, Exeter, Devon* EX4 4QE)

[Received 24 November 1998. Revise 17 May 1999]

Summary

The standard boundary-layer equations play a central role in many aspects of fluid mechanics as they describe the motion of a slightly viscous fluid close to a surface. A number of exact solutions of these equations have been found and here we extend the class of known solutions by implementing the Clarkson–Kruskal direct method (**1**) for finding similarity reductions. We demonstrate the existence of reductions which take the boundary-layer forms to lower-order partial differential systems and others which simplify them to the solution of ordinary differential equations. It is shown how not only do we recover many of the previously known exact solutions but also find some completely new forms.

1. Introduction

Almost a century ago, Prandtl realised the key part that boundary layers play in determining accurately the flow of certain fluids. He showed for slightly viscous flows that although viscosity is negligible in the bulk of the flow, it assumes a vital role near boundaries. There, under suitable conditions as discussed by Schlichting (**2**), two-dimensional flow can be approximated by the dimensionless equation

$$\psi_{yyy} + \psi_x\psi_{yy} - \psi_y\psi_{xy} - \psi_{ty} + UU_x + U_t = 0. \tag{1.1}$$

Here subscripts are used to denote partial differentiation, (x, y) denote the usual orthogonal Cartesian coordinates parallel and perpendicular to the boundary $y = 0$, t is the time and ψ denotes the streamfunction such that the velocity components of the fluid in the x- and y-directions are $u = \psi_y$ and $v = -\psi_x$ respectively. Finally, $U(x, t)$ is a given external velocity field which is such that $u(x, y, t) \to U(x, t)$ as $y \to \infty$. The boundary-layer equations are usually solved subject

[†] ⟨D. K.Ludlow@cranfield.ac.uk⟩

[‡] ⟨P. A.Clarkson@ukc.ac.uk⟩

[§] ⟨drew@maths.exeter.ac.uk⟩

to suitable conditions imposed at the wall $y = 0$ and the need to match with the far-field external velocity. Often this specifies the solution uniquely, but there are circumstances (cf. (**3**)) in which this is not so and further information is required in order to tie down the solution completely.

The quest for exact solutions of the boundary-layer equations has a long history. Blasius (**4**) used a scaling reduction to take (1.1) to a single third-order ordinary differential equation (ODE) which, when integrated numerically, provides the solution appropriate to steady flow past a flat plate at zero incidence to a uniform stream. Further work (**5** to **9**) has led to exact solutions of (1.1) corresponding to stagnation point flows, flows past wedges, jets and flows near an oscillating plate. Rayleigh (**10**) was the first to derive a solution of (1.1) relating to an unsteady flow; he demonstrated that the flow induced by an impulsively started flat surface may be obtained in terms of error functions. Jones and Watson (**11**) have given a comprehensive account of many of the classical exact solutions of the boundary-layer equations including Falkner–Skan forms and the asymptotic suction profile.

Our objective in this article is to find 'similarity reductions' of (1.1); by this we mean solutions of the partial differential equation (PDE) which may either be expressed in terms of a lower-order PDE or an ODE. We conduct a systematic investigation into the variety of reductions possible and do so by applying the so-called direct method of Clarkson and Kruskal (**1**) to (1.1). The essential idea of this procedure is to seek solutions in the form

$$\psi(x, y, t) = F\left(x, y, t, w(\eta, \zeta)\right), \tag{1.2}$$

where $\eta = \eta(x, y, t)$, $\zeta = \zeta(x, y, t)$ and F and w are sufficiently differentiable functions of their respective arguments. The substitution of (1.2) in (1.1) and the requirement that w satisfies either a PDE with fewer independent variables or an ODE imposes conditions on the functions in the form of an overdetermined system of equations whose solution leads to the desired reductions. It is not difficult to show that $F_{ww} = 0$, without loss of generality, and that one of the variables ζ, η may be taken as independent of y (cf. (**12**)). Consequently, it is sufficient to look for solutions of (1.1) in the special form

$$\psi(x, y, t) \equiv \alpha(x, y, t) + \beta(x, y, t)w\left(\eta(x, y, t), \zeta(x, t)\right), \tag{1.3}$$

where $\beta \not\equiv 0$. It is noted that there are three freedoms implicit within the ansatz (1.3)

F1 : α may be translated by a function of the form $\beta\Gamma(\eta, \zeta)$; (1.4a)

F2 : β may be scaled by any function of η and ζ; and (1.4b)

F3 : freedom in the precise functional forms of η and ζ. (1.4c)

We shall make extensive use of the freedoms F1 to F3 in order to simplify the calculations as much as possible. The freedoms F1 and F2, of translation and scaling, may be applied *once* only, without loss of generality, during the calculation for each dependent symmetry variable, and freedom F3 may also be applied *once* only, without loss of generality. In this context it is worth remarking that the nature of this work means that it is impossible to give comprehensive details of every stage of each calculation. Our aim is to provide a complete account of the outcome of applying the direct method to the boundary-layer equations while avoiding the intricate details of the complicated manipulations that are inevitably involved. An extended discussion of the calculations may be found in the first author's thesis (**13**).

Variants of the direct method have been applied to several problems within fluid mechanics but no previous work has attempted to use the full version as proposed in (**1**). Williams and Johnson

(**14**) applied a simplified scheme to the unsteady boundary-layer equations and thereby derived a low-order PDE which was solved numerically. Burdé (**15**) studied the steady equations and retrieved some of those previously known solutions which were mentioned above, together with some new forms relevant to flows over permeable surfaces. He later extended his work (**16** to **18**) and reduced the boundary-layer equations to systems of ODEs: several new explicit solutions of boundary-layer problems (especially axisymmetric solutions) were found and some of these appear undetected by other similarity reduction methods. Weidman and Amberg (**19**) applied Burdé's ideas to thermal laminar convective flows past heated plates and found that their new solutions fell into two categories: one class corresponds to flows within a wedge while the other pertains to rectilinear flows over flat plates.

We have already indicated that our concern here is restricted to the classical two-dimensional unsteady boundary-layer equation (1.1) and this paper is the first to consider solutions arising from the complete direct method ansatz (1.3). We remark that Burdé generated his similarity solutions of several physically interesting flows with a restricted ansatz which consists of a proper subset of the forms satisfying (1.3). Clearly it would be of interest to apply the full form of (1.3) to his problems but this is left to future studies. Instead, here we organize the rest of the work as follows. First, in section 2, we review quickly the classical similarity reductions for (1.1). Then, section 3 contains reductions of (1.1) to PDEs in two independent variables and in section 4 we examine similarity forms which reduce the system to scalar ODEs. The remaining reductions of (1.1) which may be accessed via the direct method are considered in section 5 and the paper is rounded off in section 6 with a short discussion. Throughout we make frequent contact with previously derived similarity solutions and show that while the direct method recovers many of these forms it also finds several completely new ones.

2. Classical similarity reductions

The classical method for finding similarity reductions (or symmetry reductions) of PDEs is the Lie-group method of infinitesimal transformations (cf. (**20** to **28**)). Ovsiannikov (**26**) gives a comprehensive account of the application of the classical method to the boundary-layer equations and, in particular, notes that a variety of outcomes are possible depending upon one's view of the task at hand. The essence of the Lie-group method is that each of the variables in the initial equation is subjected to an infinitesimal transformation (explained further below) and the demand that the equation is invariant under these transformations leads to the determination of the possible symmetries. Now this technique can be routinely applied to the boundary-layer equation (1.1) so that the external velocity field $U(x, t)$ is treated as a variable and subject to transformation. However, as Ovsiannikov points out, in hydrodynamical problems it is most often the case that $U(x, t)$ (or, equivalently the fluid pressure in the boundary layer) is prescribed. Moreover the group analyses of the problems differ depending on whether $U(x, t)$ is treated as a variable or not. As part of the rationale to this work is to seek new similarity solutions that may have physical meaning, we are persuaded to conduct the Lie-group method under the auspices that $U(x, t)$ is assumed. The external velocity is most easily eliminated from (1.1) by taking its y-derivative. Then, to apply the classical method to

$$(\psi_{yyy} + \psi_x\psi_{yy} - \psi_y\psi_{xy} - \psi_{ty})_y = 0, \tag{2.1}$$

we consider the one-parameter Lie group of infinitesimal transformations in (x, y, t, ψ) given by

$$\begin{aligned} \tilde{x} &= x + \varepsilon\xi(x, y, t, \psi) + O(\varepsilon^2), \\ \tilde{y} &= y + \varepsilon\eta(x, y, t, \psi) + O(\varepsilon^2), \\ \tilde{t} &= t + \varepsilon\tau(x, y, t, \psi) + O(\varepsilon^2), \\ \tilde{u} &= u + \varepsilon\phi(x, y, t, \psi) + O(\varepsilon^2), \end{aligned} \tag{2.2}$$

where ε is the group parameter. Requiring that (2.1) is invariant under this transformation yields an overdetermined, linear system of equations for the infinitesimals $\xi(x, y, t, \psi)$, $\eta(x, y, t, \psi)$, $\tau(x, y, t, \psi)$ and $\phi(x, y, t, \psi)$. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$\mathbf{v} = \xi(x, y, t, \psi)\frac{\partial}{\partial x} + \eta(x, y, t, \psi)\frac{\partial}{\partial y} + \tau(x, y, t, \psi)\frac{\partial}{\partial t} + \phi(x, y, t, \psi)\frac{\partial}{\partial u}. \tag{2.3}$$

Though this method is entirely algorithmic, it often involves a large amount of tedious algebra and auxiliary calculations which can become virtually unmanageable if attempted manually. Symbolic manipulation programs have been developed to facilitate the calculations. These programs use packages including MACSYMA, MAPLE, MATHEMATICA and REDUCE: an excellent survey of the different codes presently available and a discussion of their strengths and applications is given by Hereman (**29**).

Applying the classical method to (2.1) yields the infinitesimals

$$\begin{aligned} \xi(x, y, t, \psi) &= (3c_1 + c_2)x + g(t), \\ \eta(x, y, t, \psi) &= c_1 y + \frac{\partial f}{\partial x}, \\ \tau(x, y, t, \psi) &= 2c_1 t + c_3, \\ \phi(x, y, t, \psi) &= (2c_1 + c_2)\psi + y\frac{dg}{dt} - \frac{\partial f}{\partial t}, \end{aligned} \tag{2.4}$$

where c_1, c_2 and c_3 are arbitrary constants and $f(x, t)$ and $g(t)$ are sufficiently differentiable arbitrary functions. The infinitesimals (2.4) have been derived by Ma and Hui (**30**) and Rogers and Ames (**27**) and Ovsiannikov (**26**) gives a comprehensive account of the differences that arise in the infinitesimals should $U(x, t)$ be treated as an unknown at the outset. Of these differences, perhaps the most remarkable is that the full Lie algebra of the system becomes infinite dimensional. However, returning to our study here, when $U(x, t)$ is prescribed the vector fields $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_5$ associated with (2.4) are given by

$$\begin{aligned} &\mathbf{v}_1 = 3x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2t\frac{\partial}{\partial t} + 2\psi\frac{\partial}{\partial \psi}, \qquad \mathbf{v}_2 = x\frac{\partial}{\partial x} + \psi\frac{\partial}{\partial \psi}, \qquad \mathbf{v}_3 = \frac{\partial}{\partial t}, \\ &\mathbf{v}_4 = g(t)\frac{\partial}{\partial x} + y\frac{dg}{dt}\frac{\partial}{\partial \psi}, \qquad \mathbf{v}_5 = \frac{\partial f}{\partial x}\frac{\partial}{\partial y} - \frac{\partial f}{\partial t}\frac{\partial}{\partial \psi} \end{aligned} \tag{2.5}$$

and the corresponding invariant transformations are

$$(x, y, t, \psi) \longmapsto \left(e^{3\varepsilon}x, e^{\varepsilon}y, e^{2\varepsilon}t, e^{2\varepsilon}\psi\right), \tag{2.6a}$$

$$(x, y, t, \psi) \longmapsto \left(e^{\varepsilon}x, y, t, e^{\varepsilon}\psi\right), \tag{2.6b}$$

$$(x, y, t, \psi) \longmapsto (x, y, t+\varepsilon, \psi), \tag{2.6c}$$

$$(x, y, t, \psi) \longmapsto \left(x + \varepsilon g(t), y, t, \psi + \varepsilon y \frac{dg}{dt}\right), \tag{2.6d}$$

$$(x, y, t, \psi) \longmapsto \left(x, ay + \varepsilon \frac{\partial f}{\partial x}, t, \psi - \varepsilon \frac{\partial f}{\partial t}\right). \tag{2.6e}$$

These transformations give freedoms in the definitions of x, y and t which are useful when deriving a similarity reduction of the boundary-layer equations (1.1) using the direct method. Further, when recording our reductions below we shall appeal to both transformations (2.6) and the freedoms F1 to F3 in order to simplify our presented results as far as is possible. Of course, it then has to be remembered that we have the potential to generalize our findings by use of some, or all, of these freedoms.

There have been several generalizations of the classical Lie-group method for symmetry reductions. Ovsiannikov (**26**) developed the method of partially invariant solutions; recently Ondich (**31**) has shown that this method can be considered as a special case of the method of differential constraints introduced by Yanenko (**32**) and Olver and Rosenau (**33**, **34**). Bluman and Cole (**35**), in their study of symmetry reductions of the linear heat equation, proposed the so-called non-classical method of group-invariant solutions. Subsequently, these methods were further generalized by Olver and Rosenau (**33**, **34**) to include 'weak symmetries' and, even more generally, 'side conditions' or 'differential constraints' (see also (**32**)). However, their framework appears to be too general to be practical.

Motivated by the fact that symmetry reductions of the Boussinesq equation were known that are not obtainable using the classical Lie-group method (cf. (**33**, **34**)), Clarkson and Kruskal (**1**) developed an algorithmic method for finding symmetry reductions (the *direct method* introduced in section 1), which they used to obtain previously unknown reductions of the Boussinesq equation. Levi and Winternitz (**36**) subsequently gave a group-theoretical explanation of the results of Clarkson and Kruskal by showing that all the new reductions of the Boussinesq equation could be obtained using the non-classical method of Bluman and Cole (**35**). The novel characteristic of the direct method, in comparison to the others mentioned above, is that it involves no use of group theory. We remark that the direct method has certain resemblances to the so-called 'method of free parameter analysis' (cf. (**37**)); though in this latter method the boundary conditions are crucially used in the determination of the reduction whereas they are not used in the direct method. Additional ansatz-based methods for determining reductions and exact solutions of PDEs have been used by Fushchych and co-workers (cf. (**38** to **40**) and the references therein).

The non-classical method lay dormant for several years, essentially until the papers by Olver and Rosenau (**33**, **34**); in fact, the determining equations for the original example discussed by Bluman and Cole (**35**), namely the linear heat equation, have only been solved in general very recently by Mansfield (**41**). However, following the development of the direct method there has been renewed interest in the non-classical method and recently both these approaches have been used to generate many new symmetry reductions and exact solutions for several physically significant PDEs which represents significant and important progress (cf. (**38**, **40**, **42**, **43**) and the references therein). Recent extensions of the direct method include those due to Burdé (**16**, **17**), Galaktionov (**44**) and Hood (**45**). Generalizations of the non-classical method are discussed by Bluman and Shtelan (**46**), Burdé (**18**) and Olver and Vorob'ev (**47**).

The direct method is more general than the classical Lie method, except for implicit reductions, has no associated group framework and enables one to choose the dimension of the reduced equation. Furthermore it is a one-step procedure in as much that it can be used to reduce a PDE such as the boundary-layer equation (1.1), which has three independent variables, to an ODE in a single procedure, rather than first reducing (1.1) to a PDE with two independent variables and then reducing again to an ODE (see (**12**, **48** to **50**) for examples). However, the determining equations are nonlinear, the associated vector fields have no Lie-algebraic structure and there are only limited symbolic manipulation programs available. Recently Olver (**51**) (see also (**48**, **52** to **54**)) has discussed the precise relationship between the direct and nonclassical methods.

Similarity reductions and exact solutions have several different important applications in the context of differential equations. Since solutions of PDEs asymptotically tend to solutions of lower-dimensional equations obtained by similarity reduction, some of these special solutions will illustrate important physical phenomena. In particular, exact solutions arising from symmetry methods can often be used effectively to study properties such as asymptotics and 'blow-up' (cf. (**44**)). Furthermore, explicit solutions (such as those found by symmetry methods) can play an important role in the design and testing of numerical integrators; these solutions provide an important practical check on the accuracy and reliability of such integrators (cf. (**55**, **56**)).

3. Reductions to partial differential equations

The implementation of the Clarkson and Kruskal direct method (**1**) starts by substituting the ansatz (1.3) into the boundary-layer equation (1.1) and this yields

$$\begin{aligned}
&\beta\eta_y^3 w_{\eta\eta\eta} + \beta^2\eta_y^2\zeta_x(w_{\eta\eta}w_\zeta - w_{\eta\zeta}w_\eta) + \beta\eta_y(\beta_x\eta_y - \beta_y\eta_x)ww_{\eta\eta} - \beta\beta_y\eta_y\zeta_x ww_{\eta\zeta}\\
&\quad + \beta\zeta_x(\beta\eta_{yy} + \beta_y\eta_y)w_\eta w_\zeta + \Big[\beta^2\eta_x\eta_{yy} - \beta\beta_x\eta_y^2 + \beta(\beta_y\eta_x - \beta\eta_{xy})\eta_y\Big] w_\eta^2\\
&\quad + (\beta\beta_{yy} - \beta_y^2)\zeta_x ww_\zeta + \Big[\beta\beta_x\eta_{yy} - \beta\beta_y\eta_{xy} + (\beta_x\beta_y - \beta\beta_{xy})\eta_y + (\beta\beta_{yy} - \beta_y^2)\eta_x\Big] ww_\eta\\
&\quad + (\beta_x\beta_{yy} - \beta_{xy}\beta_y)w^2 + \Big[3\beta\eta_y\eta_{yy} + 3\beta_y\eta_y^2 - \beta\eta_y\eta_t + \beta\eta_y^2\alpha_x - \beta\eta_x\eta_y\alpha_y\Big] w_{\eta\eta}\\
&\quad - \beta\eta_y(\zeta_x\alpha_y + \zeta_t)w_{\eta\zeta} + \big[(\beta\alpha_{yy} - \beta_y\alpha_y)\zeta_x - \beta_y\zeta_t\big] w_\zeta\\
&\quad + \big[\beta\eta_{yyy} + 3\beta_y\eta_{yy} + 3\beta_{yy}\eta_y - \beta_t\eta_y - \beta\eta_{ty} - \beta_y\eta_t + (\beta\eta_{yy} + 2\beta_y\eta_y)\alpha_x\\
&\quad - (\beta_x\eta_y + \beta\eta_{xy} + \beta_y\eta_x)\alpha_y - \beta\eta_y\alpha_{xy} + \beta\eta_x\alpha_{yy}\big] w_\eta\\
&\quad + \big[\beta_{yyy} - \beta_{ty} + \beta_{yy}\alpha_x - \beta_{xy}\alpha_y - \beta_y\alpha_{xy} + \beta_x\alpha_{yy}\big] w\\
&\quad + \alpha_{yyy} + \alpha_x\alpha_{yy} - \alpha_y\alpha_{xy} - \alpha_{ty} + UU_x + U_t = 0.
\end{aligned}\tag{3.1}$$

Since the direct method proceeds by demanding that the reduced PDE for w is dependent only on ζ and η it is clear that we should insist that the ratios of the coefficients of the various terms in (3.1) must themselves be functions of ζ and η. The details of the subsequent calculations depend crucially on whether some of the coefficients in (3.1) are zero and, for ease of presentation, we will examine the various possibilities on a case-by-case basis.

3.1 *The case* $\eta_y \not\equiv 0$, $\zeta_x \not\equiv 0$

For $\eta_y \not\equiv 0$ we can divide (3.1) throughout so as to make the coefficient of the $w_{\eta\eta\eta}$ term unity. If we then denote the coefficients of the second and fourth terms in (3.1) by $\Gamma_1(\eta, \zeta)$ and $-\Gamma_2(\eta, \zeta)$ it follows that

$$\beta^2\eta_y^2\zeta_x = \beta\eta_y^3\Gamma_1, \qquad \beta\beta_y\eta_y\zeta_x = \beta\eta_y^3\Gamma_2. \tag{3.2a, b}$$

Division of these equations, integration with respect to y and use of the freedom F2 (1.4b) to scale w yields

$$\beta \equiv \theta_1(x, t). \tag{3.3}$$

If the ratio of the fifth to first terms in (3.1) is set to be $\Gamma_3(\eta, \zeta)$ we may use (3.3) to deduce that

$$\frac{\eta_{yy}}{\eta_y} = \eta_y \frac{\Gamma_3}{\Gamma_1}.$$

Two integrations show that with the liberty to scale η according to the freedom F3,

$$\eta(x, y, t) = \theta_2(x, t)\left[y + \theta_{3,x}(x, t)\right], \tag{3.4}$$

where $\theta_2 \not\equiv 0$. Equations (3.3) and (3.4) combine with the definition of Γ_1 to yield that the latter is independent of y and so $\Gamma_1 = \Gamma_1(\zeta)$. Equation (3.2a) can be integrated to give without loss of generality

$$\zeta(x, t) = \int^x \frac{\theta_2(\tilde{x}, t)}{\theta_1(\tilde{x}, t)}\, d\tilde{x} + \tau_1(t) \tag{3.5}$$

for some function $\tau_1(t)$.

Next we examine the third term in (3.1). By the previous results

$$\frac{\theta_{1,x}}{\theta_1} = \frac{\theta_2}{\theta_1}\Gamma_4(\zeta)$$

for some function Γ_4. Integration of this equation combined with (3.5) leads (without loss of generality) to $\theta_1 \equiv \theta_1(t)$, where appeal has been made to the freedom in scaling w. Similar considerations using the coefficient of ww_η in (3.1) show that we may also take $\theta_2 \equiv \theta_2(t)$. Thus, so far we have

$$\beta(t) \equiv \theta_1(t), \qquad \eta(x, y, t) = \theta_2(t)\left(y + \theta_{3,x}\right), \qquad \zeta(x, t) = \frac{\theta_2(t)[x + \tau_1(t)]}{\theta_1(t)}. \tag{3.6}$$

In order to investigate the quantity α in (1.3) it is convenient to examine the $w_{\eta\eta\eta}$ and $w_{\eta\zeta}$ terms in (3.1). If the ratio of their coefficients is $\Gamma_5(\eta, \zeta)$ we find, after rearrangement for α_y followed by integration and use of the freedom F1, that

$$\alpha(x, y, t) = -\frac{y\theta_1(t)\zeta_t}{\theta_2(t)} + \theta_4(x, t). \tag{3.7}$$

Furthermore the ratios of the w_η and $w_{\eta\eta\eta}$ terms is a function of t alone and, as we have taken $\eta_y \not\equiv 0$, $\zeta_x \not\equiv 0$ it must be constant, $-2c_1$ say, which demonstrates that

$$\frac{d\theta_1}{dt} = c_1\theta_1(t)\theta_2^2(t). \tag{3.8}$$

We observe that the ratio of coefficients of the $w_{\eta\eta}$ and $w_{\eta\eta\eta}$ terms in (3.1) is linear in y with the coefficient of y a function of t. Using (3.6) implies that this coefficient must be of the form $c_2\eta + d\gamma_1/d\zeta$ and hence it follows that

$$\left[\theta_2(t)\frac{d\theta_1}{dt} - 2\theta_1(t)\frac{d\theta_2}{dt}\right] y + \theta_1(t)\theta_2(t)\left(\theta_{4,x} - \theta_{3,xt}\right) + x\theta_{3,xx}\left[\theta_1(t)\frac{d\theta_2}{dt} - \theta_2(t)\frac{d\theta_1}{dt}\right]$$
$$+\theta_1^2(t)\theta_{3,xx}\frac{d\tau_1}{dt} = c_2\theta_1(t)\theta_2^3(t)(y+\theta_{3,x}) + \theta_1(t)\theta_2^2(t)\frac{d\gamma_1}{d\zeta}. \tag{3.9}$$

Equating the coefficients of y here and using the result in (3.8) leads, on solving the resulting ODE for θ_2, to

$$\theta_2(t) \equiv [(c_2 - c_1)t + c_3]^{-1/2} \tag{3.10}$$

so that (3.8) then gives

$$\theta_1(t) = \begin{cases} t^{c_1} & \text{if } c_2 - c_1 = 1, c_3 = 0, \\ \exp(c_1 t) & \text{if } c_2 = c_1, c_3 = 1, \end{cases} \tag{3.11}$$

after suitable scaling of t, θ_1 and θ_2. Integration of the y-independent terms in (3.9) yields

$$\theta_4(x,t) = \theta_{3,t} + \left[\tfrac{1}{2}(c_1+c_2)\theta_1(t)\theta_2(t)\zeta(x,t) - \frac{d\tau_1}{dt}\right]\theta_{3,x}, \tag{3.12}$$

where the arbitrary function of integration has been set to zero; we are at liberty to do this as any multiple of t can be added to $\psi(x,y,t)$ without affecting either the original boundary-layer equation (1.1) or its associated boundary condition $\psi_y \to U$ as $y \to \infty$. This constraint requires that

$$U(x,t) = \lim_{y\to\infty} \psi_y = \alpha_y + \beta\eta_y W(\zeta),$$

where

$$W(\zeta) \equiv \lim_{\eta\to\infty} w_\eta(\eta,\zeta).$$

Here we have assumed $\theta_2 > 0$ and, on use of freedoms F1 to F3 and classical Lie point transformations (2.6) in order to eliminate as many arbitrary functions as possible, we are left with the following reduction.

REDUCTION 1.

$$\psi(x,y,t) = \theta_1(t)w(\eta,\zeta), \tag{3.13a}$$

where

$$\eta = \theta_2(t)y, \qquad \zeta = \frac{\theta_2(t)x}{\theta_1(t)},$$

and $\theta_1(t)$ and $\theta_2(t)$ are given by (3.11) and (3.10) respectively. The function $w(\eta,\zeta)$ satisfies both $\lim_{\eta\to\infty} w_\eta(\eta,\zeta) = W(\zeta)$ and

$$w_{\eta\eta\eta} + w_\zeta w_{\eta\eta} - w_\eta w_{\eta\zeta} + \tfrac{1}{2}(c_2 - c_1)\eta w_{\eta\eta}$$
$$+ \tfrac{1}{2}\zeta(c_1+c_2)\left(w_{\eta\zeta} - \frac{dW}{d\zeta}\right) + \tfrac{1}{2}(c_2 - 3c_1)\left(w_\eta - W\right) + W\frac{dW}{d\zeta} = 0. \tag{3.13b}$$

Note that the parameter c_2 here is not arbitrary, but may take only the values $c_2 = c_1 + 1$, in which case $\theta_1(t)$ has a power-law behaviour, or $c_2 = c_1$ when this dependence is exponential (see (3.11)). Lastly, we record that the associated external velocity field is given by $U(x, t) = \theta_1(t)\theta_2(t)W(\zeta)$.

Particular cases of this reduction were obtained by Ma and Hui (**30**) who studied the boundary-layer equations using the classical Lie-group method, which we discussed in section 2. Ma and Hui (**30**) classified their reductions and, in relation to our reduction (3.13) above, their 'class IV' result corresponds to allowing $c_1 \to \infty$ while their 'class V' arises from setting $c_1 = 0$ and using the scaling invariance (2.6d); the corresponding PDE is just the time-independent boundary-layer equation for which solutions may be deduced from some reductions discussed in section 4 below. The 'class VI' reduction found in (**30**) is also encompassed within (3.13) and when $c_1 = \frac{1}{2}$ our result is equivalent to that of Williams and Johnson (**14**) who solved the requisite PDE numerically.

Having dealt with reduced forms of (3.1) when neither η_y nor ζ_x vanishes we next examine the consequences should either function be identically zero.

3.2 *The case $\eta_y \not\equiv 0$, $\zeta_x \equiv 0$*

When $\zeta_x \equiv 0$ we can begin by noting that by invariance property (2.6e) no generality is lost by taking $\theta_3 \equiv \tau_1 \equiv 0$ in (3.6). Furthermore, we may allow $\zeta \equiv t$ and from the forms of the terms proportional to $w_{\eta\eta\eta}$ and $w_{\eta\zeta}$ in (3.1) write

$$\beta\eta_y = \beta\eta_y^3\Gamma_6^2(\eta, t). \tag{3.14}$$

Integration with respect to y yields $\Gamma_6(\eta, t) = y + \gamma_{4,x}(x, t)$ and appeal to the freedom F3 means that

$$\eta = y + \gamma_{4,x}. \tag{3.15}$$

The study of the w_ζ term in (3.1) reveals that $\beta_y = \beta\eta_y^3\Gamma_7(\eta, t)$ and division by (3.14) followed by integration shows that without loss of generality

$$\beta \equiv \theta_1(x, t). \tag{3.16}$$

A comparison of the $w_{\eta\eta\eta}$ and $w_{\eta\eta}$ coefficients in (3.1) yields, on elimination of η and β via (3.15) and (3.16)

$$\alpha_x - \gamma_{4,xx}\alpha_y = \gamma_{4,xt} + \Gamma_8(\eta, t). \tag{3.17}$$

This PDE is amenable to solution by the method of characteristics which leads to

$$\alpha(x, y, t) = \gamma_{4,t} + x\Gamma_8(\eta, t) + \phi(\eta, t) \tag{3.18}$$

for some function $\phi(\eta, t)$. The particular forms of β and η determined above also imply that the ratio of the $ww_{\eta\eta}$ (or, equivalently, the $-w_\eta^2$) and the $w_{\eta\eta\eta}$ terms in (3.1) depends on t alone. Thence $\theta_{1,x} \equiv \Gamma_9(t)$, that is,

$$\theta_1(x, t) \equiv x\Gamma_9(t) + \tau_2(t). \tag{3.19}$$

We next have to distinguish two possibilities depending on whether $\Gamma_9 \equiv 0$ or not.

3.2.1 *Sub-case* $\Gamma_9 \equiv 0$. By symmetry considerations and the freedom F1 we may safely take $\gamma_4 = 0$, $\eta = y$, $\tau_2 = 1$ and then $\beta = 1$ and, by the freedom F2, $\alpha(x, y, t) = x\Gamma_8(\eta, t)$. Substituting these values for α, β and η into the coefficients of w_η and $w_{\eta\eta\eta}$ in (3.1) shows that they are in ratio $-\Gamma_8$; it therefore only remains to force the w-independent term in (3.1) to be in the appropriate form. If this term is written $\beta\eta_y^2\Gamma_{10}(\eta, t)$ and if we define

$$U(x,t) = \lim_{y\to\infty} \psi_y(x, y, t) = \lim_{\eta\to\infty} (x\Gamma_{8,\eta} + w_\eta) \equiv xV(t) + W(t)$$

then relation (3.1) gives

$$\Gamma_{10}(\eta, t) = x\left[\Gamma_{8,\eta\eta\eta} + \Gamma_8\Gamma_{8,\eta\eta} - \left(\Gamma_{8,\eta}\right)^2 - \Gamma_{8,\eta t} + V^2(t) + \frac{dV}{dt}\right] + V(t)W(t) + \frac{dW}{dt}$$

and the coefficient of x in this expression must vanish. Consequently we have the following reduction.

REDUCTION 2.

$$\psi(x, y, t) = w(y, t) + x\Gamma_8(y, t),$$

where $w(y, t)$ and $\Gamma_8(y, t)$ satisfy the system

$$w_{yyy} + \Gamma_8 w_{yy} - w_{yt} - \Gamma_{8,y}w_y + VW + \frac{dW}{dt} = 0, \tag{3.20a}$$

$$\Gamma_{8,yyy} + \Gamma_8\Gamma_{8,yy} - (\Gamma_{8,y})^2 - \Gamma_{8,yt} + V^2 + \frac{dV}{dt} = 0, \tag{3.20b}$$

with the external velocity field given by $U(x, t) = \lim_{y\to\infty}(x\Gamma_{8,y} + w_y) \equiv xV(t) + W(t)$.

This result was obtained in (**30**) but only for the case when either w or Γ_8 vanishes; the general result (3.20) cannot be obtained using the Classical Method. Special cases of (3.20) relate to various of the more familiar exact solutions of the boundary-layer forms. For instance, when $w = 0$ and $\Gamma_{8,t} = 0$ the reduction is the solution discovered by Hiemenz (**5**) which describes flow near a forward stagnation point on a flat plate. When $\Gamma_{8,t} = 0$ and $w = e^{i\omega t}\varphi(y)$, (3.20) is equivalent to the solution of Glauert (**9**) and Rott (**57**) which is appropriate to fluid motion normal to an oscillating plate. We remark that Ma and Hui (**30**) generalized this classic solution by expressing w as a finite sum of terms of generic type $\tau(t)\varphi(y)$ and, in a similar spirit, they adapted the stagnation solution of (**5**) to generate an unsteady version which may be recovered from (3.20) by setting $w = 0$ and $\Gamma_8(y, t) = t^{-1/2} f\left(y/t^{1/2}\right)$.

3.2.2 *Sub-case* $\Gamma_9 \not\equiv 0$. This eventuality yields no new reductions for by the freedom F2 and scaling invariance (2.6d) we can take $\Gamma_9 = 1$ and $\tau_2 = 0$ so $\beta = x$. We deduce from the freedom F1 that we may put $\Gamma_8 = \gamma_4 = 0$ so that $\alpha(x, y, t) = \phi(\eta, t)$. The ratio of the $w_{\eta\eta\eta}$ and w_η terms in (3.1) implies $\phi_\eta = 0$ so that by (2.6e) $\phi = 0$. It is now easy to verify that all the terms in (3.1) have the desired forms but a simple calculation shows that all we retrieve is the special case of Reduction 2 in which $w = 0$.

The only outstanding possibility is that $\eta_y \equiv 0$ and we consider this issue next.

3.3 *The case $\eta_y \equiv 0$, $\zeta_x \not\equiv 0$*

By the freedom F3 we may take $\eta \equiv x$ and $\zeta \equiv t$ and then (3.1) reduces to

$$\begin{aligned}(\beta\beta_{yy} - \beta_y^2)ww_x + (\beta_x\beta_{yy} - \beta_{xy}\beta_y)w^2 + (\alpha_{yy}\beta - \alpha_y\beta_y)w_x - \beta_y w_t \\ + \left[\beta_{yyy} + \alpha_x\beta_{yy} - \alpha_{xy}\beta_y - \alpha_y\beta_{xy} + \alpha_{yy}\beta_x - \beta_{ty}\right] w \\ + \alpha_{yyy} + \alpha_x\alpha_{yy} - \alpha_{xy}\alpha_y - \alpha_{ty} + U_t + UU_x = 0. \quad (3.21)\end{aligned}$$

The details of the ensuing calculation sub-divide once more and depend on whether β_y vanishes. For the moment let us suppose that $\beta_y \not\equiv 0$ and if we denote the ratio of the coefficients of the ww_x and w_t terms in (3.21) as $\Gamma_{11}(x, t)$ we obtain an ODE for β with solution

$$\beta(x, y, t) = \frac{\Gamma_{11}(x, t)}{\gamma_5(x, t)} + \exp\{y\gamma_5(x, t)\}, \quad (3.22)$$

in which the coefficient of the exponential term may be set to unity by the freedom F2. If furthermore the ratio of the coefficients of w^2 and w_t is $\Gamma_{12}(x, t)$ then (3.22) yields

$$\left(\Gamma_{11,x}\gamma_5 - \Gamma_{11}(x, t)\gamma_{5,x}\right)_x - \gamma_5\gamma_{5,x}\exp\{y\gamma_5\} = \Gamma_{12}\gamma_5.$$

Equating terms independent of y reveals that $\gamma_{5,x} = 0$ so that $\gamma_5 \equiv \tau_4(t)$ and $\Gamma_{12}(x, t) = \Gamma_{11}(x, t)$. On allowing the coefficients of w_x and w_t in (3.21) to be in ratio $\Gamma_{13}(x, t)$, we can integrate the result to give

$$\alpha(x, y, t) = -y\Gamma_{13}(x, t) + \gamma_6(x, t). \quad (3.23)$$

Finally, let us suppose that the coefficients of w and w_t be related by $\Gamma_{14}(x, t)$. Examination of the result suggests that

$$\Gamma_{13}(x, t) = -\frac{x}{\tau_4(t)}\frac{d\tau_4}{dt} - \tau_5(t), \qquad \Gamma_{14}(x, t) = \tau_4^2(t) + \tau_4(t)\gamma_{6,x} - \frac{2}{\tau_4(t)}\frac{d\tau_4}{dt}.$$

All the conditions for a similarity solution are now satisfied and equation (3.21) evaluated with $w \equiv 0$ must give zero. The form of the resulting reduction can be put into generic form by noting that on redefining $\gamma_6(x, t)$ appropriately so that

$$\gamma_6(x, t) = \left[\frac{2}{\tau_4(t)}\frac{d\tau_4}{dt} - \tau_4^2(t)\right]x,$$

we can make $\Gamma_{11}(x, t) = 0$ and $w \equiv 1$: a result which follows from (2.6e). Further, we notice that by (2.6d) we may take $\tau_5(t) \equiv 0$ even if $d\tau_4/dt \equiv 0$, and what remains is the following reduction.

REDUCTION 3.

$$\psi(x, y, t) = \frac{xy}{\tau_4(t)}\frac{d\tau_4}{dt} + \left[\frac{2}{\tau_4^2(t)}\frac{d\tau_4}{dt} - \tau_4(t)\right]x + \exp\{\tau_4(t)y\}, \quad (3.24)$$

in which $\tau_4(t)$ (< 0) is an arbitrary function. The associated external velocity field is given by

$$U(x, t) = \lim_{y\to\infty}\psi_y(x, y, t) = \frac{x}{\tau_4(t)}\frac{d\tau_4}{dt}$$

and it is remarked that Ma and Hui (**30**) deduced the steady form of this solution. We note that (3.24) not only solves (1.1) but the full Navier–Stokes equations as well.

Finally, we need to comment on the possibility that $\beta_y \equiv 0$. By the freedom F2 we may take $\beta = 1$ and then if $\alpha_{yy} \not\equiv 0$ the attempted similarity solution is of no practical use since it is more difficult to find the form of the reduction than to solve the original equation! On the other hand, if $\alpha_{yy} \equiv 0$ then the trivial solution $\psi(x, y, t) = U(x, t)y + \chi(x, t)$ follows but this too is unlikely to be of much physical relevance as for this solution the tangential fluid velocity ψ_y is constant across the entire width of the boundary layer.

4. Reductions to ordinary differential equations. Case I: $z_y \not\equiv 0$

In the previous calculations we studied reductions of the boundary-layer equations (1.1) to PDEs and, in both this and the following sections, we consider reductions to ODEs. In this case the ansatz (1.3) is replaced by

$$\psi(x, y, t) = \alpha(x, y, t) + \beta(x, y, t)w\left(z(x, y, t)\right), \qquad \beta \not\equiv 0. \tag{4.1}$$

For ease of presentation the notation for arbitrary functions used above will be followed again. It should be remembered that as far as these arbitrary functions are concerned there is no relationship implied between their forms in the two entirely separate calculations, namely the cases when (I) $z_y \not\equiv 0$, which is considered here, and (II) $z_y \equiv 0$, which is examined in section 5 below.

The substitution of (4.1) into (1.1) yields

$$\begin{aligned}\beta z_y^3 \Bigg\{ \frac{d^3 w}{dz^3} + \Gamma_0 w^2 + \Gamma_1 \left[w\frac{d^2 w}{dz^2} - \left(\frac{dw}{dz}\right)^2 \right] + \Gamma_2 w \frac{dw}{dz} + \frac{d\Gamma_3}{dz}\left(\frac{dw}{dz}\right)^2 \\ + \Gamma_4 \frac{d^2 w}{dz^2} + \Gamma_5 \frac{dw}{dz} + \frac{d^2\Gamma_6}{dz^2} w + \Gamma_7 \Bigg\} = 0,\end{aligned} \tag{4.2}$$

where $\Gamma_0(z)$, $\Gamma_1(z)$, …,$\Gamma_7(z)$ are necessarily functions of z in order that w satisfies an ODE and are defined by

$$\beta_x\beta_{yy} - \beta_y\beta_{xy} = \beta z_y^3 \Gamma_0(z); \tag{4.3a}$$

$$\beta_x z_y - \beta_y z_x = z_y^2 \Gamma_1(z); \tag{4.3b}$$

$$\beta\beta_x z_{yy} + (\beta_x\beta_y - \beta\beta_{xy})z_y - \beta\beta_y z_{xy} + (\beta\beta_{yy} - \beta_y^2)z_x = \beta z_y^3 \Gamma_2(z); \tag{4.3c}$$

$$\beta(z_x z_{yy} - z_{xy} z_y) = z_y^3 \frac{d\Gamma_3}{dz}(z); \tag{4.3d}$$

$$3\beta z_{yy} + (3\beta_y + \alpha_x\beta)z_y - (\alpha_y\beta z_x + \beta z_t) = \beta z_y^2 \Gamma_4(z); \tag{4.3e}$$

$$\begin{aligned}\beta z_{yyy} + (3\beta_y + \alpha_x\beta)z_{yy} + (3\beta_{yy} + 2\alpha_x\beta_y - \alpha_y\beta_x - \beta_t - \alpha_{xy}\beta)z_y \\ -\alpha_y\beta z_{xy} + (\alpha_{yy}\beta - \alpha_y\beta_y)z_x - \beta z_{ty} - \beta_y z_t = \beta z_y^3 \Gamma_5(z);\end{aligned} \tag{4.3f}$$

$$\beta_{yyy} + \alpha_x\beta_{yy} - \alpha_{xy}\beta_y - \alpha_y\beta_{xy} + \alpha_{yy}\beta_x - \beta_{ty} = \beta z_y^3 \frac{d^2\Gamma_6}{dz^2}; \tag{4.3g}$$

$$\alpha_{yyy} + \alpha_x\alpha_{yy} - \alpha_{xy}\alpha_y - \alpha_{ty} + UU_x + U_t = \beta z_y^3 \Gamma_7(z). \tag{4.3h}$$

Here we shall take $z_y \not\equiv 0$ (the case $z_y \equiv 0$ will be discussed in section 5 below). Solving (4.3b,d) for β_x and z_{xy} reduces (4.3c) to

$$\left(2\frac{d\Gamma_3}{dz}+\Gamma_1\right)\beta_y=\left(\Gamma_2+\frac{d\Gamma_1}{dz}\right)\beta z_y. \tag{4.4}$$

We now have to distinguish between two cases.

4.1 *Solutions when* $\Gamma_1+2d\Gamma_3/dz\not\equiv 0$

When $\Gamma_1\not\equiv -2d\Gamma_3/dz$ we may integrate (4.4) and appeal to the freedom F2 to deduce that

$$\beta=\chi_1(x,t) \qquad \text{and} \qquad \Gamma_2\equiv -\frac{d\Gamma_1}{dz}. \tag{4.5}$$

As long as $\chi_{1,x}\not\equiv 0$ we may use (4.3b) and a suitable scaling based on the freedom F3 to find

$$\int^z \Gamma_1(\tilde{z})\,d\tilde{z}=\chi_{1,x}y+\chi_2(x,t), \qquad z(x,y,t)=\chi_{1,x}(y+\chi_{2,x}). \tag{4.6a, b}$$

Equation (4.3g) now reduces to an ODE for α in terms of y with x and t as parameters so that

$$\alpha(x,y,t)=\chi_3(x,t)y+\chi_4(x,t) \tag{4.7}$$

after appeal to the freedom F1. The substitution for β in (4.3d) demonstrates that $d\Gamma_3/dz$ must be constant, c_1-1 say, and what remains is an ODE for χ_1:

$$\frac{\chi_{1,xx}}{\chi_{1,x}}=(1-c_1)\frac{\chi_{1,x}}{\chi_1}. \tag{4.8}$$

4.1.1 *Solutions when* $c_1\neq 0$. Suppose that $c_1=1/q$; equation (4.8) then gives $\chi_1=\tau_1(t)[x+\tau_2(t)]^q$. The expressions for α, β and z when inserted into (4.3e,f) determine $\Gamma_4=c_2z+c_3$, $\Gamma_5=c_4$ for constants c_i, $i=2,3,4$. After splitting the first of these equations into coefficients of y we obtain

$$\left[\tau_1(t)\chi_{3,x}-\frac{d\tau_1}{dt}\right][x+\tau_2(t)]+(1-q)\tau_1(t)\left[\chi_3(x,t)+\frac{d\tau_2}{dt}\right]-c_2q^2\tau_1^3(t)[x+\tau_2(t)]^{2q-1}=0, \tag{4.9a}$$

$$\begin{aligned}&\left[\tau_1(t)\chi_{4,x}-\tau_1(t)\chi_3(x,t)\chi_{2,xx}-\frac{d\tau_1}{dt}\chi_{2,x}-\tau_1(t)\chi_{2,xt}\right][x+\tau_2(t)]-c_3q\tau_1^2(t)[x+\tau_2(t)]^q\\&\quad+(1-q)\tau_1(t)\chi_{2,x}\left[\chi_3(x,t)+\frac{d\tau_2}{dt}\right]-c_2q^2\tau_1^3(t)\chi_{2,x}[x+\tau_2(t)]^{2q-1}=0,\end{aligned} \tag{4.9b}$$

$$\left[\tau_1(t)\chi_{3,x}+2\frac{d\tau_1}{dt}\right][x+\tau_2(t)]+(2q-1)\tau_1(t)\left[\chi_3(x,t)+\frac{d\tau_2}{dt}\right]+c_4q^2\tau_1^3(t)[x+\tau_2(t)]^{2q-1}=0. \tag{4.9c}$$

On solving for χ_3 from (4.9a to c) it transpires that as long as $q\neq 1$ or $q\neq \frac{2}{3}$ we may scale $\tau_1=1$, $c_4=\chi_3=0$ by use of the freedoms F1 and F2. Equations (4.3a to h) are now all satisfied provided that $\Gamma_7=(2-1/q)W(2c_2+W)$, where $W=\lim_{z\to\infty}dw/dz$. After a little rescaling and simplification using the classical transformations (2.6) we have the following reduction.

REDUCTION 4.

$$\psi(x, y) = x^q w(z), \tag{4.10a}$$

where $z(x, y) \equiv x^{q-1}y$ and $w(z)$ satisfies

$$\frac{d^3w}{dz^3} + qw\frac{d^2w}{dz^2} + (1-2q)\left\{\left(\frac{dw}{dz}\right)^2 - W^2\right\} = 0 \tag{4.10b}$$

subject to $\lim_{z\to\infty} dw/dz = W$.

This reduction can be generalized using the classical Lie-group transformations (2.6). In particular, use of (2.6d,e) yields

$$\psi(x, y, t) = [x + g(t)]^q w(z) - \left\{y + \frac{\partial f}{\partial x}(x, t)\right\}\frac{dg}{dt}(t) + \frac{\partial f}{\partial t}(x, t),$$

where $z(x, y, t) \equiv [x + g(t)]^{q-1}[y + f_x(x, t)]$, $f(x, t)$ and $g(t)$ are arbitrary functions, and $w(z)$ satisfies (4.10b). We remark that all of the reductions below can be enriched using these classical Lie-group transformations, though we leave this to the reader.

The solution (4.10) corresponds to an external flow velocity $U(x) = x^{2q-1}W$ and, while its form is steady, of course time dependence can be introduced using various of (2.6). Moreover, this structure is precisely the similarity solution of Falkner and Skan (**6**) and when solved subject to $w(0) = w'(0) = 0$, with $' \equiv d/dz$, it yields the flow past a wedge of arbitrary angle. More particularly, when $q = \frac{1}{2}$ we retrieve Blasius's equation and for $q = \frac{1}{3}$ we obtain an equation derived in (**8**): this can be integrated twice to give a Ricatti equation which is linearizable by setting $w(z) = 6v'(z)/v(z)$. The function $v(z)$ can be found in terms of Airy and parabolic cylinder functions (see Abramowitz and Stegun (**58**)) and the resulting solution describes the flow produced by a fine jet emerging into a quiescent fluid. Recently similarity solutions of the steady boundary-layer equations yielding special cases of (4.10) have also been discussed in (**59**, **60**).

Interest is also aroused in the special case $q = 2$ and $W = 0$ for then the resulting ODE is the so-called Chazy equation whose general solution may be expressed in terms of hypergeometric functions (**61**). It is of importance because it is the simplest example of an ODE whose solutions possess a natural movable boundary; that is, a closed curve in the complex plane beyond which a solution cannot be continued analytically. The Chazy equation has a wide spectrum of applications including number theory and solitons; more details may be found in (**62**, **63**) and the references therein.

For over a century workers have been investigating under what circumstances ODEs may be solved exactly, either by using standard techniques or by appeal to Lie-group theory. Experience has suggested that explicit solutions of an ODE can only usually be obtained if the ODE is of so-called *Painlevé type*: that is, if the only movable singularities in any solution are poles. Ablowitz *et al.* (**64**) developed a method for determining whether an equation is of Painlevé type and we can apply this test—the so-called Painlevé test—to equation (4.10b). The conditions demanded by the test are only satisfied for the special parameter values mentioned in the preceding paragraph. Therefore analytical solution is most unlikely for other parameters and then one has to resort to numerical solution.

In the special case $q = \frac{2}{3}$ a more general solution of (4.9) may be obtained. On appeal to the inherent properties of the freedoms F1 to F3 we may put $\tau_1 \equiv 1$, $c_4 = -c_2$ and

$$\chi_3 = \tfrac{2}{3}c_2[x + \tau_2(t)]^{1/3} + c_5[x + \tau_2(t)]^{-1/3} - \frac{d\tau_2}{dt}.$$

Equations (4.3) all hold provided $\Gamma_7 = c_2 W + \frac{1}{2}W^2$ and, after removing superfluous functions by appropriate scaling, we obtain the following reduction.

REDUCTION 5.

$$\psi(x, y) = x^{2/3}w(z) + \lambda_2 x^{-1/3}y, \tag{4.11a}$$

where $z(x, y) \equiv x^{-1/3}y$ and $w(z)$ satisfies

$$\frac{d^3w}{dz^3} + \tfrac{2}{3}w\frac{d^2w}{dz^2} - \tfrac{1}{3}\left(\frac{dw}{dz}\right)^2 + \tfrac{1}{3}W^2 = 0, \tag{4.11b}$$

subject to $\lim_{z\to\infty} dw/dz = W$, a constant.

Now the external velocity $U(x)$ is equal to $x^{1/3}W + \lambda_2 x^{-1/3}$ and when $\lambda_2 = 0$ this form is a special case of Reduction 4; however when $\lambda_2 \not\equiv 0$ Reduction 5 is not classical. This result was not obtained by Burdé (**15**) owing to the restricted nature of his initial ansatz and so it appears to be a novel reduction of (1.1) despite the fact that the corresponding ODE is closely related to the Falkner–Skan equation. Moreover, equation (4.11b) is not of Painlevé type so that realizable analytical solutions are not likely.

Lastly we examine the choice $q = 1$. Equations (4.9a to c) lead to

$$\frac{d\tau_1}{dt} = -\tfrac{1}{4}(c_4 + 2c_2)\tau_1^3(t), \qquad \chi_3(x, t) = -\tfrac{1}{4}(c_4 - 2c_2)[x + \tau_2(t)]\tau_1^2(t) - \frac{d\tau_2}{dt}.$$

If $\tau_1 \equiv 0$ we recover Reduction 4 while if $\tau_1 \not\equiv 0$ we obtain $\tau_1(t) = t^{-1/2}$ and $c_4 = 2(1 - c_2)$ using the freedom F2 and the scaling transformation (2.6c). All the conditions for a similarity reduction are satisfied provided that $\Gamma_7 = W^2 + 2(1 + c_2)W$ and after setting $\lambda_1 = c_2 - \frac{1}{2}$ we obtain the following reduction.

REDUCTION 6.

$$\psi(x, y, t) = xt^{-1/2}w(z), \tag{4.12a}$$

where $z(y, t) \equiv yt^{-1/2}$ and $w(z)$ satisfies

$$\frac{d^3w}{dz^3} + w\frac{d^2w}{dz^2} - \left(\frac{dw}{dz}\right)^2 + \tfrac{1}{2}z\frac{d^2w}{dz^2} + \frac{dw}{dz} + W^2 - W = 0, \tag{4.12b}$$

with $\lim_{z\to\infty} dw/dz = W$.

We note that this reduction was obtained by Ma and Hui (**30**) and corresponds to an external velocity distribution $U(x, t) = xt^{-1}W$. With $w(0) = dw(0)/dz = 0$, $W = 1$ this particular solution is a generalized version of the PDE Reduction 3.

4.1.2 *Solutions when* $c_1 = 0$. We next have to re-examine the solution of (4.8) when $c_1 = 0$. From the equations corresponding to (4.9a,c) we discover that we may scale $\tau_1 = 1$, $c_4 = -4c_2$ and $\chi_3 = c_2 \exp(2x)$ and thence follows the following reduction.

REDUCTION 7.

$$\psi(x, y, t) = w(z)\exp(x), \tag{4.13a}$$

where $z(x, y) \equiv y\exp(x)$ and $w(z)$ satisfies both $\lim_{z\to\infty} dw/dz = W$ and

$$\frac{d^3w}{dz^3} + w\frac{d^2w}{dz^2} - 2\left(\frac{dw}{dz}\right)^2 + 2W^2 = 0. \tag{4.13b}$$

This well-known reduction of the steady boundary-layer equations, which has an associated external velocity field given by $U(x) = W\exp(2x)$, is a limiting case of the Falkner–Skan solutions (**6**).

4.1.3 *Solutions for* $\chi_{1,x} \equiv 0$. A further special case arises when $\chi_{1,x} \equiv 0$ for then $\beta = \tau_1(t)$ by (4.5) and governing equations (4.3a,b,c,g) are trivially satisfied provided that $\Gamma_0 = \Gamma_1 = \Gamma_2 = \Gamma_6 = 0$. Integration of (4.3d) yields the implicit solution

$$\Theta_1(z, t) = \Gamma_3(z)x - \chi_3(x, t) - \tau_1(t), \tag{4.14}$$

and, since $2d\Gamma_3/dz + \Gamma_1 \not\equiv 0$ by assumption, the freedom F3 means that we may take $\Gamma_3 \equiv z$. Rearrangements of (4.3d,e) give expressions for z_{xy} and α_x which when substituted in (4.3f) yield a form for α_y. The compatibility condition $(\alpha_x)_y = (\alpha_y)_x$ together with differential consequences of the result

$$z_y = \frac{\tau_1(t)}{x - \Theta_{1,z}}, \tag{4.15}$$

which follows directly from (4.14), simplify to show that

$$\frac{\tau_1^2(t)}{(x-\Theta_{1,z})^4}\left[12\Theta_{1,zz}^2 + (5\Theta_{1,zzz} - \Gamma_4\Theta_{1,zz})(x-\Theta_{1,z}) - \frac{d\Gamma_4}{dz}(x-\Theta_{1,z})^2 + \frac{2(x-\Theta_{1,z})^4}{\tau_1^3(t)}\frac{d\tau_1}{dt}\right] = 0$$

for some function $\Gamma_4(z)$. It may be deduced that $d\tau_1/dt = 0$ so that we scale $\tau_1 \equiv 1$, $\Gamma_4 = c_5$, a constant, and $\Theta_1 \equiv -\tau_2(t)z + \tau_4(t)$. We then find from (4.15) that $z(x, y, t) \equiv (y + \chi_{2,x})/[x + \tau_2(t)]$ which in turn gives

$$\alpha(x, y, t) = -\frac{d\tau_2}{dt}y + \chi_{2,t} - \frac{d\tau_2}{dt}\chi_{2,x} + c_5\ln[x + \tau_2(t)].$$

These results when combined and simplified using the transformations (2.6) lead to the following reduction.

REDUCTION 8.

$$\psi(x, y) = w(z) + c_5 \ln x, \tag{4.16a}$$

where $z(x, y) \equiv y/x$ and $w(z)$ satisfies

$$\frac{d^3w}{dz^3} + \left(\frac{dw}{dz}\right)^2 + c_5\frac{d^2w}{dz^2} - W^2 = 0 \tag{4.16b}$$

together with $\lim_{z\to\infty} dw/dz = W$ so that $u(x) \to U(x) = W/x$ as $y \to \infty$. Equation (4.16b) is of Painlevé type if and only if $c_5 = 0$ or $c_5^2 = 25W^2/3$: in either case it may be solved in terms of appropriate Weierstrass elliptic functions.

4.2 *Solutions when* $\Gamma_1 + 2d\Gamma_3/dz \equiv 0$

Having completed our analysis for $\Gamma_1 + 2d\Gamma_3 dz \not\equiv 0$ we now revert to (4.4) when equality does hold. Then $\Gamma_2 = -d\Gamma_1/dz$ and the elimination of $d\Gamma_3/dz$ between (4.3b,d) gives a first-order PDE for β with solution of the form

$$\beta(x, y, t) = \Theta_2(x, t)z_y^2. \tag{4.17}$$

We proceed by using (4.3d) which facilitates the elimination of z_{xy} and higher differential consequences of z in favour of z_x and y-derivatives of z. Derivatives of α with respect to x can be removed between equations (4.3e,f) and, assuming $d\Gamma_3(z)/dz \not\equiv 0$, it is possible to solve algebraic equations for α_y. The substitution of this solution into (4.3e) yields an expression for α_x and use of the compatibility constraint $\alpha_{xy} = \alpha_{yx}$ leads to a form which when integrated gives $y + \chi_{2,x}(x, t) = \Theta_3(z, t)$. An application of the implicit-function theorem tells us that

$$z(x, y, t) = \Phi_1(y + \chi_{2,x}, t) \text{ and hence } \beta = \Phi_2(y + \chi_{2,x}, t) \tag{4.18}$$

for some functions $\Phi_1(s, t)$ and $\Phi_2(s, t)$, to be determined. Substituting back into (4.3d) implies that $d\Gamma_3/dz = 0$ which provides a contradiction unless $\Theta_{2,t} \equiv 0$. Hence, on making use of the freedom F2, $\beta = z_y^2$.

A repetition of this procedure with $\Theta_2 \equiv 1$ leads to $z(x, y, t) = \chi_1^{1/2}(y + \chi_{2,x})$ and $\beta = \chi_1$. Equation (4.3d) implies that $d\Gamma_3/dz$ is constant and integration combined with the available scalings give $\chi_1 = [x + \tau_2(t)]^2$. Comparison of these forms for β, z and χ_1 with those obtained earlier in (4.5), (4.6b) and following reveals that this case will yield nothing new and in fact will duplicate Reduction 4, equation (4.10), with $q = 2$.

4.3 *Solutions when* $d\Gamma_3/dz = \Gamma_1 = \Gamma_2 \equiv 0$

Last in this section we address the almost degenerate case when $d\Gamma_3/dz = \Gamma_1 = \Gamma_2 \equiv 0$. From (4.18) above $z(x, y, t) = \Phi_1(p, t)$ and $\beta = \Phi_2(p, t)$ where $p = y + \chi_{2,x}$. If for convenience we also put $\alpha \equiv \Xi(p, x, t) + \chi_{2,t}$, substitution of these forms for α, β and z in (4.3a to h) yields four trivial equations and four more-involved relations. The solution of (4.3e) for Ξ gives

$$\Xi(p, x, t) = \left(\frac{\Phi_{1,t}}{\Phi_{1,p}} + \Gamma_4\Phi_{1,p} - 3\frac{\Phi_{2,p}}{\Phi_2} - 3\frac{\Phi_{1,pp}}{\Phi_{1,p}}\right)x + \Phi_4(p, t)$$

which suggests that rather than obtaining the expected reduction to an ODE only a PDE will result. A rather tedious manipulation yields a final result which takes precisely the same form as Reduction 2 above and nothing new arises.

5. Reductions to ordinary differential equations. Case II: $z_y \equiv 0$

The reductions discussed in the previous section were derived using (4.1) with $z_y \not\equiv 0$. For the sake of completeness we now next examine reductions for which z_y does vanish and it is convenient to examine these separately from those with $z_y \not\equiv 0$ largely because the form of the governing equations is fundamentally altered by inclusion of this extra restriction. Now substitution of (4.1) in (1.1) yields not (4.2) but rather

$$\begin{aligned}\left(\beta\beta_{yy} - \beta_y^2\right) z_x w \frac{dw}{dz} + \left(\beta_x\beta_{yy} - \beta_{xy}\beta_y\right) w^2 + \left[\left(\beta\alpha_{yy} - \beta_y\alpha_y\right) z_x - \beta_y z_t\right] \frac{dw}{dz}\\ + \left[\beta_{yyy} + \beta_{yy}\alpha_x - \beta_y\alpha_{xy} - \beta_{xy}\alpha_y + \beta_x\alpha_{yy} - \beta_{ty}\right] w\\ + \alpha_{yyy} + \alpha_x\alpha_{yy} - \alpha_{xy}\alpha_y - \alpha_{ty} + UU_x + U_t = 0. \quad (5.1)\end{aligned}$$

It is natural to impose the restriction on ansatz (4.1) that $\beta_y \to 0$ as $y \to \infty$ for otherwise the limiting velocity U $(= \lim_{y\to\infty} \psi_y)$ is a function of w. However, it is also important to realise that nothing will be lost by this assumption, because this restriction on β can be compensated for by the addition of appropriate terms to α. The principal attraction of this imposition lies in the observation that with it the far-field external velocity profile depends only on the y-derivative of α.

Recall that the aim is to force the ratios of coefficients in (5.1) to be functions of z and we commence our discussion by assuming that $(\beta\beta_{yy} - \beta_y^2)z_x \not\equiv 0$. Then if we denote the ratio of the first two coefficients in (5.1) by $\Gamma_8(z)$, two integrations and the freedom F2 to scale w leads to

$$\Gamma_8 = 0, \qquad \beta = \Phi_{3,p}(p,t), \quad (5.2)$$

where $p \equiv y + \chi_{2,x}(x,t)$. Further, if the ratio of the dw/dz and wdw/dz coefficients is $\Gamma_9(z)$, integrations with respect to y and the freedom F1 imply that

$$\Gamma_9 = 0, \qquad \alpha(x,y,t) = \chi_5(x,t)\Phi_3(p,t) - p\frac{z_t}{z_x} + \chi_6(x,t) + \chi_{2,t}. \quad (5.3)$$

If we then demand that the remaining coefficients in (5.1) are of the appropriate forms we obtain the two equations

$$\begin{aligned}\Phi_{3,pppp} - \frac{d\Gamma_{10}}{dz} z_x(\Phi_{3,p}\Phi_{3,ppp} - \Phi_{3,pp}^2) + \chi_{5,x}(\Phi_3\Phi_{3,ppp} - \Phi_{3,p}\Phi_{3,pp})\\ + \left(\frac{z_t}{z_x}\right)_x (\Phi_{3,pp} - p\Phi_{3,ppp}) + \chi_{6,x}\Phi_{3,ppp} - \Phi_{3,ppt} = 0, \quad (5.4a)\end{aligned}$$

$$\begin{aligned}\chi_5\Phi_{3,ppp} + \chi_5\chi_{5,x}(\Phi_3\Phi_{3,pp} - \Phi_{3,p}^2) - \frac{d\Gamma_{11}}{dz} z_x(\Phi_{3,p}\Phi_{3,ppp} - \Phi_{3,pp}^2)\\ + \chi_5\left(\frac{z_t}{z_x}\right)_x (\Phi_{3,p} - p\Phi_{3,pp})\\ + \chi_5\chi_{6,x}\Phi_{3,pp} - \chi_5\Phi_{3,pt} + \left(\chi_{5,x}\frac{z_t}{z_x} - \chi_{5,t}\right)\Phi_{3,p} - \frac{z_t}{z_x}\left(\frac{z_t}{z_x}\right)_x + \left(\frac{z_t}{z_x}\right)_t\\ + UU_x + U_t = 0 \quad (5.4b)\end{aligned}$$

for some functions $\Gamma_{10}(z)$ and $\Gamma_{11}(z)$. Although these equations are somewhat complicated, they can be solved routinely using the following method. If χ_5 multiples of (5.4a) are subtracted from the p-derivative of (5.4b) we are left with

$$\left\{\frac{\chi_{5,x} z_t}{z_x} - \chi_5\left(\frac{z_t}{z_x}\right)_x - \chi_{5,t}\right\}\Phi_{3,pp} + \frac{d\Gamma_{10}}{dz}\chi_5 z_x\left(\Phi_{3,p}\Phi_{3,ppp} - \Phi_{3,pp}^2\right) + \frac{d\Gamma_{11}}{dz} z_x(\Phi_{3,pp}\Phi_{3,ppp} - \Phi_{3,p}\Phi_{3,pppp}) = 0. \quad (5.5)$$

On division through by $\Phi_{3,p}^2$ followed by two integrations with respect to p, we find that

$$\frac{d\Gamma_{11}}{dz}\Phi_{3,pp} - \frac{d\Gamma_{10}}{dz}\chi_5\Phi_{3,p} - \chi_7(x,t)\Phi_3 = \left\{\chi_{5,t} - \frac{\chi_{5,x} z_t}{z_x} + \chi_5\left(\frac{z_t}{z_x}\right)_x\right\}\frac{p}{z_x} + \chi_8(x,t),$$

which can be easily solved to yield

$$\Phi_3(p,t) = A_+(t)\exp\{m_+(t)p\} + A_-(t)\exp\{m_-(t)p\} + \frac{dB}{dt}p + D(t). \quad (5.6)$$

Here $A_\pm(t)$, $B(t)$ and $D(t)$ are arbitrary functions; moreover the exponents $m_\pm(t) < 0$ (except possibly when $\chi_5 = 0$ and the term independent of w in (5.1) vanishes). This result follows from the imposition that $\lim_{p\to\infty}\Phi_{3,pp} = 0$ and, using the form we now have for ψ, it is apparent that

$$U(x,t) = \frac{dB}{dt}\chi_5(x,t) - \frac{z_t}{z_x}.$$

Substituting both this result and (5.6) back into (5.4) yields sets of equations which relate the coefficients in (5.6). It is safe to take $A_+(t) \neq 0$ and then

$$z_t = \left\{x\frac{dB}{dt}\chi_5(x,t) + m_+(t)\frac{d}{dt}\left[\frac{\tau_2(t)}{m_+(t)}\right]\right\}z_x, \quad (5.7a)$$

$$\chi_6(x,t) \equiv \Gamma_{10}(z)\frac{dB}{dt} - D(t)\chi_5(x,t) + x\left[\frac{3}{m_+^2(t)}\frac{dm_+}{dt} + \frac{1}{A_+(t)m_+(t)}\frac{dA_+}{dt} - m_+(t)\right], \quad (5.7b)$$

and

$$A_+(t)m_+^2(t)z_x\frac{dB}{dt}\left[\chi_5(x,t)\frac{d\Gamma_{10}}{dz} - m_+(t)\frac{d\Gamma_{11}}{dz}\right] + A_+(t)m_+(t)\left(\frac{d\tau_2}{dt}\chi_{5,x} - \chi_{5,t}\right) + A_+(t)\frac{dm_+}{dt}\left\{\chi_5(x,t) - \chi_{5,x}[x + \tau_2(t)]\right\} = 0. \quad (5.8)$$

Let us first suppose that $A_-(t) \equiv 0$ which forces $dB/dt \equiv 0$ or else our original supposition $\beta\beta_{yy} \not\equiv \beta_y^2$ is violated. In order to obtain a similarity reduction we have to solve (5.7a) and (5.8) subject to χ_6 given by (5.7b). Equation (5.7a) is solved using the method of characteristics which yields, after an application of the implicit-function theorem,

$$z = \zeta(\xi,\tau), \qquad \chi_5 = \frac{m_+(t)\zeta_\tau}{B'(t)\zeta_\xi},$$

where $\xi = [x + \tau_2(t)]/m_+(t)$, $\zeta_\xi \not\equiv 0$ and $\tau = t$. Substitution into (5.7b) gives an equation for ζ from which we obtain the following reduction.

REDUCTION 9.

$$\psi(x, y, t) = A_+(t)m_+(t)\exp\{m_+(t)y\}\left[\frac{\zeta_t}{B'(t)\zeta_\xi} + w(\zeta)\right] + \frac{dB}{dt}[w(\zeta) + \Gamma_{10}(\zeta)]$$
$$+\left\{\frac{3}{m_+(t)}\frac{dm_+}{dt} + \frac{1}{A_+(t)}\frac{dA_+}{dt} - m_+^2(t) + y\frac{dm_+}{dt}\right\}\frac{x}{m_+(t)}, \tag{5.9}$$

where $m_+(t)$, $A_+(t)$ and $B(t)$ are arbitrary functions, $\mathrm{Re}(m_+(t)) < 0$, $\xi = x/m_+(t)$ and $\zeta = \zeta(\xi, t)$ satisfies

$$\frac{\zeta_t}{B'(t)\zeta_\xi}\frac{\partial}{\partial t}\left(\frac{\zeta_t}{B'(t)\zeta_\xi}\right) - \frac{\zeta_t^2}{B'(t)\zeta_\xi}\frac{d\Gamma_{10}}{d\zeta} + \zeta_t\frac{d\Gamma_{11}}{d\zeta} = 0$$

for arbitrary functions $\Gamma_{10}(\zeta)$ and $\Gamma_{11}(\zeta)$ while $w(\zeta)$ satisfies

$$w\frac{dw}{d\zeta} + \frac{d\Gamma_{10}}{d\zeta}w + \frac{d\Gamma_{11}}{d\zeta} = 0.$$

Note that for this solution the external flow velocity is

$$U(x, t) = \lim_{y\to\infty}\psi_y(x, y, t) = \frac{x}{m_+(t)}\frac{dm_+}{dt}.$$

Now let us reconsider the problem when $A_-(t) \not\equiv 0$. Without loss of generality it follows that $m_+(t) \not\equiv m_-(t)$ and neither vanishes: in fact

$$c_6 m_-(t) = m_+(t), \qquad \text{where} \quad c_6 \neq 0, 1 \tag{5.10}$$

and use of (5.4b) leads to $\chi_5(x, t) = m_+(t)\gamma_1(z)$ and $d\Gamma_{11}/dz = c_6\gamma_1 d\gamma_1/dz$, where $\gamma_1(z)$ is to be found. (We are not assuming that $\chi_{5,x} \neq 0$ here so are not at liberty to invoke the freedom F2.) Substituting for χ_5 in (5.4a) implies that $d\Gamma_{10}/dz = (1 + c_6)d\gamma_1/dz$ and, given this result for χ_5 and (5.7b) for χ_6, (5.4a) yields an equation relating $A_+(t)$, $A_-(t)$ and $m_+(t)$ which integrates once to

$$A_+(t) = c_7[A_-(t)]^{c_6}[m_+(t)]^{3(c_6-1)}\exp\left\{\frac{c_6 - 1}{c_6}\int^t m_+^2(\tilde{t})\,d\tilde{t}\right\}, \tag{5.11}$$

with $c_7 \neq 0$. We remark that (5.8) is now identically satisfied and so it just remains to integrate (5.7a) by characteristics to give

$$\gamma_2(z) - B\gamma_1(z) = \frac{x + \tau_2(t)}{m_+(t)}. \tag{5.12}$$

If $dB/dt \not\equiv 0$ then, after simplification, we obtain the following reduction.

REDUCTION 10.

$$\psi(x, y, t) = \left[A_+(t)m_+(t)\exp\{m_+(t)y\} + \frac{A_-(t)m_+(t)}{c_6}\exp\{m_+(t)y/c_6\} + \frac{dB}{dt}\right]w(z)$$
$$+\left[A_+(t)m_+(t)\exp\{m_+(t)y\} + A_-(t)m_+(t)\exp\{m_+(t)y/c_6\} + \frac{dB}{dt}(1 + c_6)\right]\gamma_1(z)$$
$$+\left[\frac{3}{m_+(t)}\frac{dm_+}{dt} + \frac{1}{A_+(t)}\frac{dA_+}{dt} - m_+^2(t) + y\frac{dm_+}{dt}\right]\frac{x}{m_+(t)}, \tag{5.13a}$$

where $A_-(t)$ and $m_+(t)$ (< 0) are arbitrary functions, $A_+(t)$ is defined by (5.11), z is a function of $x/m_+(t)$, $B(t)$ defined implicitly by (5.12) for arbitrary $\gamma_1(z)$ and $\gamma_2(z)$ and $w(z)$ satisfies the ODE

$$w\frac{dw}{dz} + (1 + c_6)\frac{d\gamma_1}{dz}w + c_6\gamma_1\frac{d\gamma_1}{dz} = 0. \tag{5.13b}$$

We notice that (5.13b) admits the obvious solution $w(z) = -\gamma_1(z)$.

This reduction is associated with the corresponding external flow

$$U(x, t) = \frac{x}{m_+(t)}\frac{dm_+}{dt}$$

and, lastly, we remark that should $dB/dt \equiv 0$ then exactly the same reduction can be derived.

Assuming that $\beta\beta_{yy} \neq \beta_y^2$, the only possibility not covered thus far arises should the function χ_5 in (5.3) vanish. The terms proportional to $\exp\{[m_+(t) + m_-(t)]/p\}$ in (5.4b) inform us that $d\Gamma_{11}/dz = 0$ and, by differentiating (5.4a) once it follows that

$$(\Gamma_{10})_{xx} = \left(\frac{z_t}{z_x}\right)_{xx}\frac{\Phi_{3,pp} - p\Phi_{3,ppp}}{\Phi_{3,p}\Phi_{3,ppp} - \Phi_{3,p}^2} + \chi_{6,xx}\frac{\Phi_{3,ppp}}{\Phi_{3,p}\Phi_{3,ppp} - \Phi_{3,pp}^2}.$$

Appeal is now made to the fact that the right-hand side is a function of x and t alone and we can deduce, by forcing the respective functions of p and t to be independent of p, that Φ_3 is precisely of form (5.6) unless

$$\left(\frac{z_t}{z_x}\right)_{xx} = \chi_{6,xx} = (\Gamma_{10})_{xx} = 0.$$

Then $z(x, t) = [x + \tau_2(t)]/\tau_5(t)$, $d\Gamma_{10}/dz = -c_8$, $\chi_6(x, t) = \tau_6(t)[x + \tau_2(t)]$ and $w(z)$ satisfies

$$w(z)\left(\frac{dw}{dz} - c_8\right) = 0.$$

Thus $w(z) = c_8 z + c_9$, with c_8 and c_9 arbitrary constants. For a non-trivial reduction we require at least one of $c_8, c_9 \neq 0$ and then, after suitable relabelling, we obtain the following reduction.

REDUCTION 11.

$$\psi(x, y, t) = \frac{xy}{\tau_5(t)}\frac{d\tau_5}{dt} + x\tau_6(t) + (c_8 z + c_9)\Phi_4(y, t), \tag{5.14a}$$

where $\tau_5(t)$ and $\tau_6(t)$ are arbitrary, $z(x, t) = x/\tau_5(t)$ and $\Phi_4(y, t)$ satisfies both $\lim_{y\to\infty}\Phi_{4,y} = 0$ and the PDE

$$\Phi_{4,yyy} + \frac{c_8(\Phi_4\Phi_{4,yy} - \Phi_{4,y}^2)}{\tau_5(t)} - \frac{\Phi_{4,y} - y\Phi_{4,yy}}{\tau_5(t)}\frac{d\tau_5}{dt} + \tau_6(t)\Phi_{4,yy} - \Phi_{4,yt} = 0. \tag{5.14b}$$

It is clear that the associated external velocity field is

$$U(x, t) = \frac{x}{\tau_5(t)}\frac{d\tau_5}{dt}.$$

Reductions 9 to 11 appear to be novel and not recorded previously in the literature. They constitute the set of similarity forms that arise from (5.1) when the leading-order term in that equation is not identically zero; for completeness we next need to examine the situation when this term does vanish.

5.1 *Reductions arising when* $\beta\beta_{yy} = \beta_y^2$, $\beta_y \neq 0$

The constraint $\beta\beta_{yy} = \beta_y^2$ implies that $\beta = \exp\{\chi_9(x,t)p\}$, where $p = y + \chi_{2,x}$ and we shall assume for the meantime that $\chi_{9,x} \not\equiv 0$ so that the coefficient of w^2 in (5.1) is non-zero. If we denote the ratio of the dw/dz and w^2 coefficients in (5.1) by $-\Gamma_{12}(z)$, then integrating gives

$$\alpha(x,y,t) = -\frac{z_t p}{z_x} + \left[\frac{\chi_{9,x}\Gamma_{12}(z)p}{z_x} + \chi_{11}\right]\exp(\chi_9 p) + \chi_{10}(x,t). \tag{5.15}$$

The coefficients of $y\exp\{2\chi_9(x,t)p\}$ and $\exp\{2\chi_9(x,t)p\}$ in (5.1) give that $\Gamma_{12} = 0$ and the freedom F3 means that we can put $\chi_{11} = 0$ whence $U = -z_t/z_x$. Further equations are determined by supposing that the ratios of the coefficients of w and w^0 with that of w^2 in (5.1) are $\Gamma_{13}(z)$ and $\Gamma_{14}(z)$. After substitution for α and β it can be deduced that χ_{11} in (5.15) is a function of z and, given the nature of (5.15) and the form of β, the freedom F1 enables us to set $\chi_{11} = \Gamma_{13} = \Gamma_{14} = 0$, without loss of generality. The resulting ODE is actually the trivial $w = 0$ and so we have the simple exact solution

$$\psi(x,y,t) = U(x,t)p + \chi_{10}(x,t).$$

If, however, we take $\chi_{9,x} = 0$ then a more interesting solution appears. For now $\beta = \exp\{\tau_7(t)p\}$, where $\tau_7(t) < 0$ and, for convenience, we let $\alpha(x,y,t) = \Xi(p,x,t) + \chi_{2,t}$ and assume that $(\alpha_{pp} - \tau_7\alpha_p)z_x \not\equiv \tau_7 z_t$. If the ratio of the coefficients of the w and dw/dz terms in (5.1) is now written as $d\Gamma_{15}/dz$ we obtain a PDE for $\Xi(p,x,t)$ which, after suitable scaling for which we may take $\Gamma_{15} = 0$, admits the solution

$$\Xi(p,x,t) = \left[\frac{p}{\tau_7(t)}\frac{d\tau_7}{dt} + 2\tau_7^2(t)\frac{d\tau_7}{dt} - \tau_7(t)\right]x + \chi_{11}(x,t)\exp\{\tau_7(t)p\} + \Phi_5(p,t) \tag{5.16}$$

for some function $\Phi_5(p,t)$. We deduce that

$$U(x,t) = \frac{x}{\tau_7(t)}\frac{d\tau_7}{dt} + V(t),$$

where $V(t) = \lim_{p\to\infty}\Phi_{5,p}(p,t)$. Inspection of the w^2 and w-independent terms in (5.1) yields $dw/dz = -d\Gamma_{16}/dz$ for some function Γ_{16} which may be equated to zero as this function of z can be incorporated into χ_{11}. Therefore the w-independent terms in (5.1) must vanish and this requirement gives rise to a PDE for χ_{11} which may be solved by the method of characteristics to give

$$\begin{aligned}\chi_{11}(x,t) = \int^t \Bigg\{&\Phi_{5,ppp} + \left[\frac{p}{\tau_7^2(\tilde{t})}\frac{d\tau_7}{d\tilde{t}} + \frac{2}{\tau_7(\tilde{t})}\frac{d\tau_7}{d\tilde{t}} - \tau_7(\tilde{t})\right]\Phi_{5,pp} + \frac{V(\tilde{t}) - \Phi_{5,p}}{\tau_7(\tilde{t})}\frac{d\tau_7}{d\tilde{t}} \\ &+ \frac{dV}{d\tilde{t}} - \Phi_{5,p\tilde{t}}\Bigg\}\frac{\exp\{-\tau_7(\tilde{t})p\}}{\tau_7(\tilde{t})}\,d\tilde{t} + f(s),\end{aligned} \tag{5.17}$$

with

$$s = \frac{x}{\tau_7(t)} + \int^t \frac{\Phi_{5,pp} - \tau_7(\tilde{t})\Phi_{5,p}}{\tau_7^2(\tilde{t})}\,d\tilde{t}$$

and where $f(s)$ is an arbitrary function. Two distinguished possibilities now arise.

5.1.1 *Solutions when* $f_{xx} \equiv 0$. When $f_{xx} \equiv 0$ then $\chi_{11}(x,t) = c_{10}x/\tau_7(t)$ without loss of generality so that we have the following reduction.

REDUCTION 12.

$$\psi(x,y,t) = \left[\frac{y}{\tau_7(t)}\frac{d\tau_7}{dt} + \frac{2}{\tau_7^2(t)}\frac{d\tau_7}{dt} - \tau_7(t)\right]x + \frac{c_{10}x}{\tau_7(t)}\exp\{\tau_7(t)y\} + \Phi_5(y,t), \tag{5.18a}$$

where $\tau_7(t)(<0)$ is an arbitrary function, c_{10} is a constant and $\Phi_5(y,t)$ satisfies

$$\Phi_{5,yyy} + \left[\frac{y}{\tau_7(t)}\frac{d\tau_7}{dt} + \frac{2}{\tau_7^2(t)}\frac{d\tau_7}{dt} - \tau_7(t)\right]\Phi_{5,yy} + [V(t) - \Phi_{5,y}]\tau_7(t)\frac{d\tau_7}{dt} + \frac{dV}{dt}$$
$$- \Phi_{5,yt} + \frac{c_{10}\exp\{\tau_7(t)y\}}{\tau_7(t)}[\Phi_{5,yy} - \tau_7(t)\Phi_{5,y}] = 0, \tag{5.18b}$$

with $V(t) = \lim_{y\to\infty}\Phi_{5,y}$. Then

$$U(x,t) = \lim_{y\to\infty}\psi_y = \frac{x}{\tau_7(t)}\frac{d\tau_7}{dt} + V(t).$$

5.1.2 *Solutions when* $f_{xx} \not\equiv 0$. In this circumstance it is necessary that $\Phi_{5,pp} - \tau_7(t)\Phi_{5,p}$ is purely a function of t and hence $\Phi_5 = V(t)p$ (the exponential and p-independent terms can be freely set to zero). After suitable redefinitions according to the transformations (2.6), this form of Φ_5 yields the exact solution, as follows.

REDUCTION 13.

$$\psi(x,y,t) = \frac{xy}{\tau_7(t)}\frac{d\tau_7}{dt} + \left[\frac{2}{\tau_7^2(t)}\frac{d\tau_7}{dt} - \tau_7(t)\right]x + f(s)\exp\{\tau_7(t)y\}, \tag{5.19}$$

where $s = x/\tau_7(t)$ and $\tau_7(t)$ (<0) and $f(s)$ are arbitrary functions.

5.2 *Solutions for* $(\alpha_{pp} - \tau_7(t)\alpha_p)z_x \equiv \tau_7(t)z_t$

When α satisfies this identity the analysis outlined above requires some modification. Rather than $\alpha(x,y,t) = \Xi(p,x,t) + \chi_{2,t}$ with Ξ given by (5.16), now

$$\alpha(x,y,t) = -(z_t p/z_x) + \chi_{13}\exp\{\tau_7(t)p\} + \chi_{12} + \chi_{2,t}$$

and $U = -z_t/z_x$. If the ratio of the coefficients of the w-independent and w terms in (5.1) is denoted by $-\Gamma_{17}$ (assuming the latter coefficient does not vanish) we discover that on substituting for α and β then should $\chi_{13} \equiv -\Gamma_{17}$ (and so may be assumed zero by the freedom F2) then w satisfies the degenerate ODE $w = 0$. On the other hand, if $\chi_{13} \not\equiv -\Gamma_{17}$ it follows from the defining equations for Γ_{17} that $z(x,t) = [x + \tau_2(t)]/\tau_7(t)$ and

$$\chi_{12,x} = \frac{\chi_{13,x}}{\tau_7^2(t)(\Gamma_{17}+\chi_{13})}\left\{[x+\tau_2(t)]\frac{d\tau_7}{dt} - \tau_7(t)\frac{d\tau_2}{dt}\right\} + \frac{\tau_7(t)\chi_{13,t}}{\tau_7^2(t)(\Gamma_{17}+\chi_{13})} + \frac{2}{\tau_7^2(t)}\frac{d\tau_7}{dt} - \tau_7(t).$$

Further, w now satisfies $w \equiv \Gamma_{17}$ but this function can be combined with χ_{13} and thus w reduced to zero. Then we have the following reduction.

REDUCTION 14.

$$\psi(x,y,t) = \frac{xy}{\tau_7(t)}\frac{d\tau_7}{dt} + \chi_{13}(x,t)\exp\{\tau_7(t)y\} + \left[\frac{2}{\tau_7^2(t)}\frac{d\tau_7}{dt} - \tau_7(t)\right]x$$
$$+\frac{1}{\tau_7^2(t)}\int^t \left\{x\chi_{13,x}\frac{d\tau_7}{d\tilde{t}} + \tau_7(\tilde{t})\chi_{13,\tilde{t}}\right\}\frac{d\tilde{t}}{\chi_{13}(x,\tilde{t})}, \tag{5.20}$$

where $\tau_7(t)$ (< 0) and $\chi_{13}(x,t)$ are arbitrary functions. When $\chi_{13} = f(z)$, with $z(x,t) = x/\tau_7(t)$, this reduction coincides with (5.19) but, even when this particular relationship is not operational, both (5.19) and (5.20) are associated with the same external velocity field

$$U(x,t) = \frac{x}{\tau_7(t)}\frac{d\tau_7}{dt}.$$

5.3 *Other cases*

To finish, all that is left is to address the possibility that all the coefficients in (5.1) might vanish. In order that all the coefficients save those of the w and w-independent terms be zero it has already been shown that

$$\alpha(x,y,t) = -z_t p/z_x + \chi_{13}\exp\{\tau_7(t)p\} + \chi_{12} + \chi_{2,t}, \qquad \beta = \exp\{\tau_7(t)p\}.$$

The coefficient of w disappears if

$$z(x,y) = \frac{x+\tau_2(t)}{\tau_7(t)}, \qquad \chi_{12}(x,t) = \left[\frac{2}{\tau_7^2(t)}\frac{d\tau_7}{dt} - \tau_7(t)\right][x+\tau_2(t)]$$

and consideration of the terms independent of w just leads to $\chi_{13} \equiv \gamma(z) \equiv 0$ by appeal to the freedom F2. The remaining exact solution is precisely Reduction 13, equation (5.19).

Two final results arise should $\beta_y \equiv 0$. If $z_y \equiv 0$, $z_x \not\equiv 0$ then if $\alpha_{yy} \equiv 0$ we merely obtain the trivial solution $\psi(x,y,t) = U(x,t)y + \chi_{14}(x,t)$. Conversely, if $\alpha_{yy} \not\equiv 0$ it is straightforward to demonstrate that by the freedom F2 we may pick $\beta_x \equiv 0$ and we are left with the task of insisting

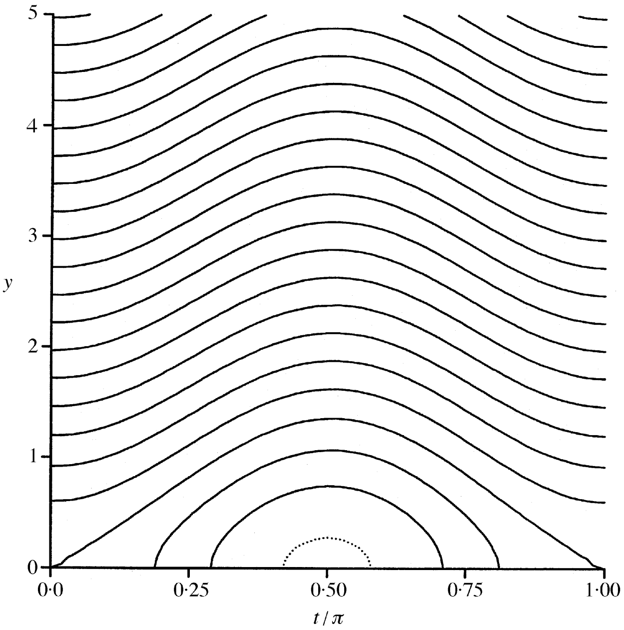
$$\left\{\alpha_{yyy} + \alpha_x\alpha_{yy} - \alpha_{xy}\alpha_y - \alpha_{ty} + UU_x + U_t\right\} / \left[\alpha_{yy}\beta(t)z_x\right]$$

is a function of z. However, this problem is actually no easier than directly solving the original equation so this reduction is of little use.

Finally, when $z_x \equiv z_y \equiv 0$ we appeal to the freedom F3 to set $z \equiv t$ and (5.1) dictates that

$$-\beta_y\frac{dw}{dz} + (\beta_x\beta_{yy} - \beta_{xy}\beta_y)w^2 + (\beta_{yyy} + \beta_{yy}\alpha_x - \beta_y\alpha_{xy} - \beta_{xy}\alpha_y + \beta_x\alpha_{yy} - \beta_{ty})w$$
$$+\alpha_{yyy} + \alpha_x\alpha_{yy} - \alpha_{xy}\alpha_y - \alpha_{ty} + UU_x + U_t = 0$$

must be an ODE for $w = w(t)$. For a reduction to be of any help we must have $\beta_y \not\equiv 0$: however, this degenerate case is difficult to solve.



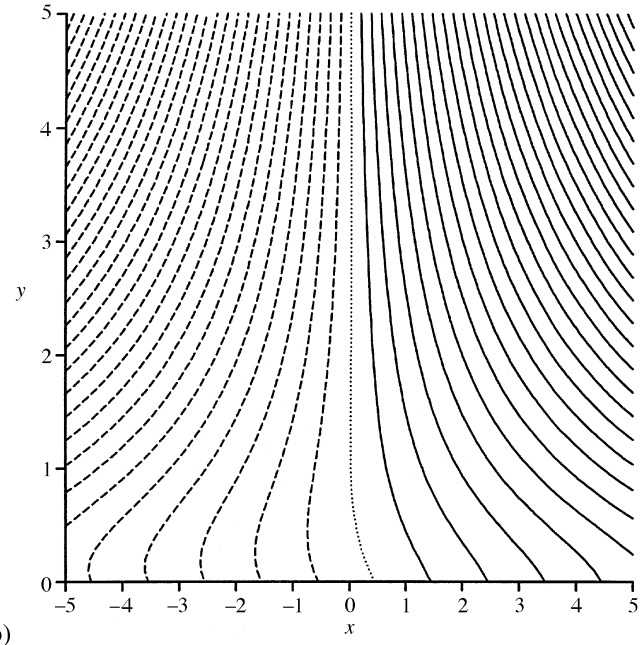
(a)

Fig. 1 Plots of the velocity components $v(y, t)$ and streamfunction $\psi(x, y, t)$ given by the solution of system (3.20) subject to $\Gamma_8 = V_{\text{suct}} = \frac{1}{5}(1 + 4\cos^2 t)$, $\Gamma_{8,y} = 0$ and $w_y = \cos t$ on $y = 0$ together with $\Gamma_{8,y} \to 1$ and $w_y \to 0$ as $y \to \infty$. (a) Contour plot of $v(y, t)$ for $0 \leqslant t \leqslant \pi$, $0 \leqslant y \leqslant 5$. This function is π-periodic in t and is negative (corresponding to velocities towards $y = 0$) at all points. There is a spacing of $\frac{1}{4}$ between successive contours and the contour $v(y, t) = -\frac{1}{4}$ is shown dotted for reference. (b) Contour plot of $\psi(x, y, t)$ as a function of coordinates x and y at time $t = 0$. The interval between successive contours is unity and regions of $\psi > 0$ ($\psi < 0$) are denoted by solid (dashed) lines. (The zero contour is shown dotted.) (c), (d) as (b) except for times $t = \frac{1}{2}\pi$ and $t = \frac{3}{4}\pi$ respectively.

6. Concluding remarks

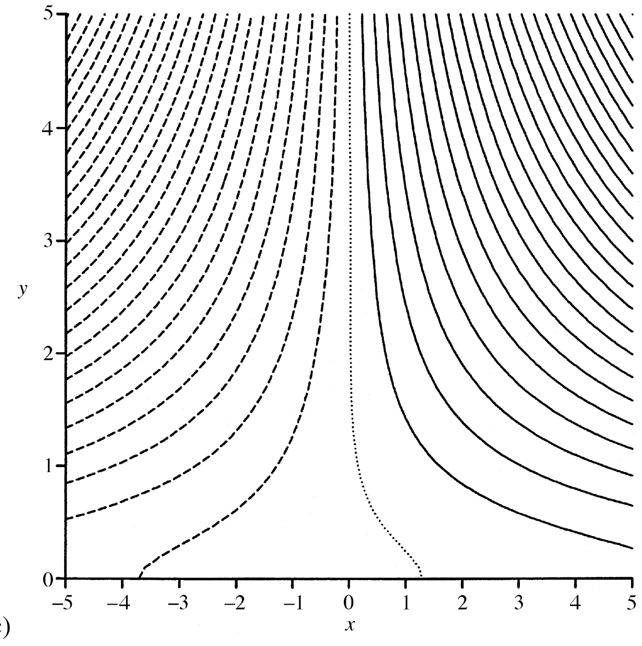
In this work we have presented a comprehensive account of the forms of similarity reductions that can arise from the two-dimensional unsteady Prandtl boundary-layer equations. In contrast to previous studies of this topic we have implemented a systematic investigation using the full form of the direct method as developed in (**1**). Other authors have typically used either other methods (for example, the classical Lie-group approach) or started with a restricted ansatz for the structure of the solution. Our work has led to a spectrum of reductions which have been catalogued in sections 3 to 5; it is worth repeating that the reductions as quoted are given in their most simplified forms. Any of them can of course be enriched by appeal to the five invariances listed in (2.6).

Explicit solutions of the boundary-layer equations are important from both theoretical and practical viewpoints. They are relevant to many flows of real concern and, even if particular solutions do not have any obvious direct application, they are frequently used as a basis of numerical or experimental studies. It is reassuring that our reductions discussed above retrieve a number of the classical physical solutions, recover some previously discovered solutions (most notably solutions given by Ma and Hui (**30**) and Burdé (**15** to **18**)) and encompass some completely new exact



(b)

Fig. 1 continued



(c)

Fig. 1 continued

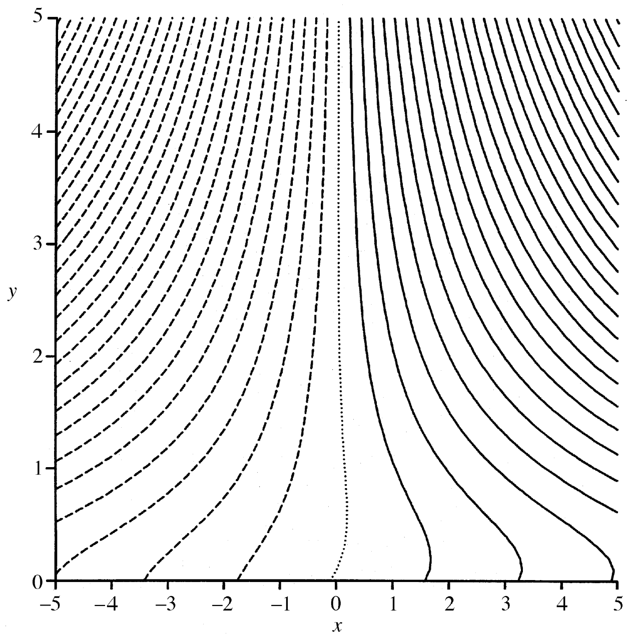
(d) 

Fig. 1 continued

solutions. Indeed, none of the reductions noted in section 5 appear to have been obtained before. One might ask whether any of these are of any physical relevance whatsoever and, in view of the existence of several arbitrary functions within the definitions, it would be a great surprise if none of our novel reductions had any applications at all. Detailed physical interpretations of each of the solutions is beyond the scope of the present work, but here we take the opportunity to mention an example which is covered by one of our new reductions.

Recall our form of Reduction 2 which, for convenience, we repeat here:

$$\psi(x, y, t) = w(y, t) + x\Gamma_8(y, t),$$

where $w(y, t)$ and $\Gamma_8(y, t)$ satisfy the system

$$w_{yyy} + \Gamma_8 w_{yy} - w_{yt} - \Gamma_{8,y} w_y + VW + \frac{dW}{dt} = 0, \tag{3.20a}$$

$$\Gamma_{8,yyy} + \Gamma_8 \Gamma_{8,yy} - (\Gamma_{8,y})^2 - \Gamma_{8,yt} + V^2 + \frac{dV}{dt} = 0, \tag{3.20b}$$

with the external velocity field given by $U(x, t) = \lim_{y\to\infty}(x\Gamma_{8,y} + w_y) \equiv xV(t) + W(t)$. Previous analyses have only uncovered this reduction when either $w(y, t)$ or $\Gamma_8 = 0$. We can use this structure to generalize the solution of (**9**) appropriate to the two-dimensional flow against an infinite plate normal to the oncoming stream. If the plate is given by $y = 0$ and the y-axis positioned so that $x = 0$ marks the dividing streamline in the steady flow outside the boundary layer on the plate then $u(y, t) \to U(x, t) = x$ as $y \to \infty$. (Strictly we could include a constant of proportionality into this definition of $U(x, t)$ but in the interests of simplicity let us assume this constant is unity.) Glauert (**9**) obtained the solution to this problem when the plate is oscillated in its own plane but our form of solution enables us to account for the supplementary feature of suction. There is overwhelming evidence that all manner of boundary-layer flows can be stabilized if suction is applied to bounding surfaces (see, for example, (**65**, **66**) or many other papers) and it is used as a control mechanism in many practical situations. Suppose here the plate is oscillated with velocity $\cos t$ and simultaneously we withdraw fluid through the plate so that on $y = 0$ the normal velocity $v = -V_{\text{suct}}(t)$. Then our flow can be described by the solution of (3.20), for our velocity components

$$u(x, y, t) \equiv w_y + x\Gamma_{8,y}, \qquad v(x, y, t) \equiv -\Gamma_8,$$

require $\Gamma_8 = V_{\text{suct}}$, $\Gamma_{8,y} = 0$ and $w_y = \cos t$ on $y = 0$ together with $\Gamma_{8,y} \to 1$ and $w_y \to 0$ as $y \to \infty$.

We solved the system (3.20) numerically for the periodic suction function $V_{\text{suct}} = \frac{1}{5}(1 + 4\cos^2 t)$ and show some representative results in Fig. 1. The governing equations are parabolic in nature and a simple time-marching routine taken from the NAG suite of programs was found to be perfectly adequate for our purposes. The solution was started at $t = 0$ with some fairly arbitrary guesses for $w(y, t)$ and $\Gamma_8(y, t)$ but after a little time the solution settled to a periodic state and all our figures relate to this periodic solution. As the velocity component $v(y, t)$ is independent of x it is possible to illustrate its form for all y and t by means of a contour diagram, see Fig. 1a. Observe that $v \sim -y$ at large distances from the plate—this is a consequence of the continuity requirement but we also notice the variability in velocity due to the changing suction. Unlike $v(y, t)$, both the horizontal velocity distribution u and the streamfunction ψ itself do depend on x. Moreover, since the solution under consideration satisfies not only the boundary-layer equations but the full Navier–Stokes equations as well, some useful insight may be gained from plots of $\psi(x, y, t)$. Hence, in Fig. 1b to d we show snapshots of $\psi(x, y, t)$ at three selected times, $t = 0, \frac{1}{2}\pi$ and $\frac{3}{4}\pi$. In the first of these, corresponding to when the applied suction is greatest, the flow pattern has distinctly shifted in the positive x-direction; moreover the suction and plate velocities are equal at this time so all the streamlines cut $y = 0$ at an angle of $\frac{1}{4}\pi$. Conversely, by the time $t = \frac{1}{2}\pi$ the suction is at a minimum and the plate itself is instantaneously at rest. Thus far fewer streamlines cut $y = 0$ and those that do are normal to the axis. The last plot, figure 1d, shows a still later instant in which the plate is moving leftwards and the general flow pattern has begun to drift in that direction.

We emphasize that this form of solution of equations (3.20) is both physically sensible and remains undetected by all previous analyses into symmetry reductions of the boundary-layer equations. An investigation into the precise circumstances under which each of our new reductions may relate to physically realistic boundary-layer flows is an obvious line of further enquiry.

Our study has concentrated on solutions of the boundary-layer equations written in standard Cartesian form. Burdé (**16**) obtained several solutions of the boundary-layer forms appropriate to flows past bodies with an axis of symmetry and it would be of interest to examine whether the application of the full Clarkson–Kruskal (**1**) procedure yields further solutions.

To close we mention that it is curious how several of our similarity solutions not only fulfill the requirements of the boundary-layer equations but also satisfy the complete Navier–Stokes system as well. In work reported elsewhere we have undertaken examinations into the possible forms of similarity solutions of the unsteady incompressible Navier–Stokes equations (see (**50**, **67**)).

Acknowledgements

We are indebted to a referee for many helpful suggestions in relation to this work. Furthermore, we would like to thank Mark Ablowitz, Philip Broadbridge, Elizabeth Mansfield and Patrick Weidman for helpful discussions. The first author is grateful to the UK Science and Engineering Research Council for support through an Earmarked Studentship.

References

1. P. A. Clarkson and M. D. Kruskal, New similarity reductions of the Boussinesq equation. *J. Math. Phys.* **30** (1989) 2201–2213.
2. H. Schlichting, *Boundary Layer Theory*, 6th edition (New York, McGraw–Hill 1968).
3. K. Stewartson, Further solutions of the Falkner–Skan equation. *Proc. Camb. phil. Soc.* **50** (1954) 454–465.
4. H. Blasius, Grenzschichten in Flüssigkeiten mit kleiner Reibung. *Z. Math. U. Phys.* **56** (1908) 1–37.
5. K. Hiemenz, Die Grenzschicht an einem in den gleichförmigen Flüssigkeitsstrom eingetauchten geraden Kreiszylinder. *Dinglers J.* **326** (1911) 321–410.
6. V. M. Falkner and S. W. Skan, Solutions of the boundary-layer equations. *Phil. Mag.* (7) **12** (1931) 865–896.
7. H. Schlichting, Laminare Strahlausbreitung. *Z. angew. Math. Mech.* **13** (1933) 260–263.
8. G. Bickley, The plane jet. *Phil. Mag.* **23** (1937) 727–731.
9. M. B. Glauert, The laminar boundary layer on oscillating plates and cylinders. *J. Fluid Mech.* **1** (1956) 97–110.
10. Lord Rayleigh, On the motion of solid bodies through viscous liquids. *Phil. Mag.* (6) **21** (1911) 697–711.
11. C. W. Jones and E. J. Watson, In *Laminar Boundary Layers* (ed. L. Rosenhead; Clarendon Press, Oxford 1963) Chapter V.
12. P. A. Clarkson and P. Winternitz, Nonclassical symmetry reductions for the Kadomtsev–Petviashvili equation. *Physica* D**49** (1991) 257–272.
13. D. K. Ludlow, Similarity reductions and their applications to equations of fluid dynamics. Ph.D. Thesis, University of Exeter (1994).
14. J. C. Williams and W. D. Johnson, Semisimilar solutions to unsteady boundary-layer flows including separation. *AIAA J.* **12** (1974) 1388–1393.
15. G. I. Burdé, A class of solutions of the boundary-layer equations. *Fluid Dynamics* **25** (1990) 201–207.
16. ——, The construction of special explicit solutions of the boundary-layer equations—steady flows. *Q. Jl Mech. appl. Math* **47** (1994) 247–260.
17. ——, The construction of special explicit solutions of the boundary-layer equations—unsteady flows. *Ibid.* **48** (1995) 611–633.
18. ——, New similarity reductions of the steady-state boundary-layer equations. *J. Phys.* A: *Math. Gen.* **29** (1996) 1665–1683.
19. P. D. Weidman and M.F. Amberg, Similarity solutions for steady laminar convection along heated plates with variable oblique suction: Newtonian and Darcian fluid flow. *Q. Jl Mech. appl. Math.* **49** (1996) 373–403.
20. G. W. Bluman and J. D. Cole, *Similiarity Methods for Differential Equations*, Applied Mathematical Sciences **13** (Springer, New York 1974).

21. G. W. Bluman and S. Kumei, *Symmetries and Differential Equations*, Applied Mathematical Sciences **81** (Springer, New York 1989).

22. J. M. Hill, *Differential Equations and Group Methods for Scientists and Engineers* (CRC Press, Boca Raton 1992).

23. N. H. Ibragimov (Ed.), *CRC Handbook of Lie Groups Analysis of Differential Equations*. I. *Symmetries, Exact Solutions and Conservation Laws* (CRC Press, Boca Raton 1993).

24. —— (Ed.), *CRC Handbook of Lie Groups Analysis of Differential Equations*. II. *Applications in Engineering and Physical Sciences* (CRC Press, Boca Raton 1994).

25. P. J. Olver, *Applications of Lie groups to Differential Equations*, Graduate Texts in Mathematics **107**, 2nd edition (Springer, New York 1993).

26. L. V. Ovsiannikov, *Group Analysis of Differential Equations* (Academic Press, New York 1982).

27. C. Rogers and W. F. Ames, *Nonlinear Boundary Value Problems in Science and Engineering* (Academic Press, Boston 1992).

28. H. Stephani, *Differential Equations, Their Solution Using Symmetries* (University Press, Cambridge 1989).

29. W. Hereman, Symbolic software for Lie symmetry analysis. In *Lie Group Analysis of Differential Equations*. III. *New Trends in Theoretical Developments and Computational Methods* (ed. N. H. Ibragimov; CRC Press, Boca Raton 1996) Chapter XII, 367–413.

30. P. K. H. Ma and W. H. Hui, Similarity solutions of the two-dimensional unsteady boundary-layer equations. *J. Fluid Mech.* **216** (1990) 537–559.

31. J. Ondich, A differential constraints approach to partial invariance. *Europ. J. Appl. Math.* **6** (1995) 631–637.

32. N. N. Yanenko, Theory of consistency and methods of integrating systems of nonlinear partial differential equations. In *Proceedings of the Fourth All-Union Mathematics Congress* (Leningrad 1964) 247–259 (Russian).

33. P. J. Olver and P. Rosenau, The construction of special solutions to partial differential equations. *Phys. Lett.* A **114** (1986) 107–112.

34. —— and ——, Group-invariant solutions of differential equations. *SIAM J. appl. Math.* **47** (1987) 263–275.

35. G. W. Bluman and J. D. Cole, The general similarity of the heat equation. *J. Math. Mech.* **18** (1969) 1025–1042.

36. D. Levi and P. Winternitz, Nonclassical symmetry reduction: example of the Boussinesq equation. *J. Phys.* A: *Math. Gen.* **22** (1989) 2915–2924.

37. A. G. Hansen, *Similarity Analyses of Boundary Value Problems in Engineering* (Prentice–Hall, Englewood Cliffs 1964).

38. W. I. Fushchych, Conditional symmetry of the equations of mathematical physics. *Ukrain. Math. J.* **43** (1991) 1456–1470.

39. ——, Ansatz '95. *J. Nonlinear Mathematical Physics* **2** (1995) 216–235.

40. ——, W. M. Shtelen and N. I. Serov, *Symmetry Analysis and Exact Solutions of the Equations of Mathematical Physics* (Kluwer, Dordrecht 1993).

41. E. L. Mansfield, The nonclassical group analysis of the heat equation. *J. Math. Anal. Appl.* **231** (1999) 526–542.

42. P. A. Clarkson, Nonclassical symmetry reductions of the Boussinesq equation. *Chaos, Solitons & Fractals* **5** (1995) 2261–2301.

43. ——, D. K. Ludlow and T. J. Priestley, The classical, direct and nonclassical methods for symmetry reductions of nonlinear partial differential equations. *Meth. appl. Anal.* **4** (1997)

173–195.

44. V. A. Galaktionov, On new exact blow-up solutions for nonlinear heat conduction equations with source and applications. *Diff. Int. Eqns* **3** (1990) 863–874.

45. S. Hood, New exact solutions of Burgers's equation—an extension to the direct method of Clarkson and Kruskal. *J. Math. Phys.* **36** (1995) 1971–1990.

46. G. W. Bluman and V. Shtelen, Developments in similarity methods related to the pioneering work of Julian Cole. In *Mathematics is for Solving Problems* (ed. L. P. Cook, V. Roytburd and M. Turin; SIAM, Philadelphia 1996) 105–117.

47. P. J. Olver and E. M. Vorob'ev, Nonclassical and conditional symmetries. In *Lie Group Analysis of Differential Equations*. III. *New Trends in Theoretical Developments and Computational Methods* (ed. N. H. Ibragimov; CRC Press, Boca Raton 1996) Chapter X, 291–328.

48. P. A. Clarkson and S. Hood, Nonclassical symmetry reductions and exact solutions of the Zabalotskaya–Khokhlov equation. *Europ. J. appl. Math.* **3** (1992) 381–414.

49. —— and ——, New symmetry reductions and exact solutions of the Davey–Stewartson equation. I. Reductions to ordinary differential equations. *J. Math. Phys.* **35** (1994) 255–283.

50. D. K. Ludlow, P. A. Clarkson and A. P. Bassom, Nonclassical symmetry reductions of the two-dimensional incompressible Navier–Stokes equations. *Stud. appl. Math.* **103** (1999) 183–240.

51. P. J. Olver, Direct reduction and differential constraints. *Proc. R. Soc.* A **444** (1994) 509–523.

52. D. Arrigo, P. Broadbridge and J. M. Hill, Nonclassical symmetry solutions and the methods of Bluman–Cole and Clarkson–Kruskal. *J. Math. Phys* **34** (1993) 4692–4703.

53. P. A. Clarkson and T. J. Priestley, On a shallow water wave system. *Stud. appl. Math.* **101** (1998) 389–432.

54. E. Pucci, Similarity reductions of partial differential equations. *J. Phys.* A: *Math. Gen.* **25** (1992) 2631–2640.

55. W. F. Ames, F. V. Postell and E. Adams, Optimal numerical algorithms. *Appl. num. Math.* **10** (1992) 235–259.

56. Yu. I. Shokin, *The Method of Differential Approximation* (Springer, New York 1983).

57. N. Rott, Unsteady viscous flow in the vicinity of a stagnation point. *Q. appl. Math.* **13** (1956) 444–451.

58. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Tables* (Dover, New York 1964).

59. P. D. Weidman, New solutions for laminar boundary layers with cross flow. *Z. angew. Math. Phys.* **48** (1997) 341–356.

60. P. D. Weidman, D. G. Kubitschek and S. N. Brown, Boundary layer similarity flow driven by power-law shear. *Acta Mech.* **120** (1997) 199–215.

61. J. Chazy, Sur les équations différentielles du troisième ordre et d'ordre supérieur dont l'intégrale générale a ses points critiques fixes. *Acta Math.* **34** (1911) 317–385.

62. M. J. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, London Mathematical Society Lecture Notes in Mathematics **149** (University Press, Cambridge 1991).

63. P. A. Clarkson and P. J. Olver, Symmetry and the Chazy equation. *J. Diff. Eqns* **124** (1996) 225–246.

64. M. J. Ablowitz, A. Ramani and H. Segur, A connection between nonlinear evolution equations and ordinary differential equations of Painlevé type. *J. Math. Phys.* **21** (1980) 715–721.

65. R. J. Lingwood, On the effects of suction and injection on the absolute instability of the rotating-disk boundary layer. *Phys. Fluids* **9** (1997) 1317–1328.

66. M. S. Mughal, Active control of wave instabilities in three-dimensional compressible flows. *Theoret. & Comput. Fluid Dyn.* **12** (1998) 195–217.

67. D. K. Ludlow, P. A. Clarkson and A. P. Bassom, Nonclassical symmetry reductions of the three-dimensional incompressible Navier–Stokes equations. *J. Phys.* A: *Math. Gen.* **31** (1998) 7965–7980.