Complete sets of invariants for dynamical systems that admit a separation of variables

E. G. Kalnins and J. M. Kress
Department of Mathematics, University of Waikato,
Hamilton, New Zealand,
e.kalnins@waikato.ac.nz and *jonathan@math.waikato.ac.nz*

G. Pogosyan
Laboratory of Theoretical Physics, Joint Institute for Nuclear Research,
Dubna, Moscow Region, 14980, Russia,
pogosyan@thsun1.jinr.dubna.su

and W. Miller, Jr.
School of Mathematics, University of Minnesota,
Minneapolis, Minnesota, 55455, U.S.A.,
miller@ima.umn.edu

February 28, 2002

Abstract

Consider a classical Hamiltonian H in n dimensions consisting of a kinetic energy term plus a potential. If the associated Hamilton-Jacobi equation admits an orthogonal separation of variables, then it is possible to generate algorithmically a canonical basis $\mathbf{Q}, \mathbf{P}$ where $P_1 = H, P_2, \cdots, P_n$ are the other 2nd-order constants of the motion associated with the separable coordinates, and $\{Q_i, Q_j\} = \{P_i, P_j\} = 0$, $\{Q_i, P_j\} = \delta_{ij}$. The $2n-1$ functions $Q_2, \cdots, Q_n, P_1, \cdots, P_n$ form a basis for the invariants. We show how to determine for exactly which spaces and potentials the invariant Q_j is a polynomial in the original momenta. We shed light on the general question of exactly when the Hamiltonian admits a constant of the motion that is polynomial in the momenta. For $n = 2$ we go further and consider all cases where the Hamilton-Jacobi equation admits a 2nd-order constant of the motion, not necessarily associated with orthogonal separable coordinates, or even separable coordinates at all. In each of these cases we construct an additional constant of the motion.

1 Introduction

The quest for integrable systems has a long history. Basically, the question is, given a classical Hamiltonian $H = H(x,p)$ where $\mathbf{x} = (x_1, ..., x_n)$, $\mathbf{p} = (p_1, ..., p_n)$ how can one find all the solutions to the Poisson bracket condition

$$\{H, L\} = \sum_{i=1}^{n}(\frac{\partial H}{\partial p_i}\frac{\partial L}{\partial x_i} - \frac{\partial H}{\partial x_i}\frac{\partial L}{\partial p_i}) = 0 \qquad (1)$$

where $L = L(\mathbf{x}, \mathbf{p})$,[1]. There is no known comprehensive solution to this problem. However, if the associated Hamilton-Jacobi equation $H(\mathbf{x}, \frac{\partial S}{\partial \mathbf{x}}) = E$ is additively separable in the orthogonal variables $\mathbf{x}$ then a complete integral of the equation can be constructed by quadratures and one can find a basis of $2n-1$ functionally independent solutions to equation (1). Indeed there is an explicit canonical change of coordinates from the variables $\mathbf{x}, \mathbf{p}$ with $\{x_i, p_j\} = \delta_{ij}$ to variables $\mathbf{Q}, \mathbf{P}$ where $P_1 = H$, $P_2, \cdots, P_n$ are the other 2nd-order constants of the motion associated with the orthogonal separable x-coordinates, and $\{Q_i, Q_j\} = \{P_i, P_j\} = 0$, $\{Q_i, P_j\} = \delta_{ij}$. Thus the $2n-1$ functions $Q_2, \cdots, Q_n, P_1, \cdots, P_n$ form a basis for the invariants. Each invariant Q_j can be expressed as a sum of the form

$$Q_j = \sum_{k=1}^{n} M_j(x_j, \mathbf{P}), \qquad (2)$$

see [1].

Numerous examples have been found through this approach, but important problems remain. Many of the known interesting dynamical systems have extra constants of the motion L which are *polynomial* in the canonical momenta $p_i, i = 1, \cdots, n$. This often enables global statements to be made about the system in question, e.g., the existence of closed orbits. However, though many interesting results have been obtained, e.g. [2], an algorithmic way of generating all polynomial solutions to (1) is not known. In particular from the x-based integrals in (2) it is difficult to tell if Q_j is a polynomial in the momenta p_i. In this article we adopt a p-based approach to the calculation of the invariants Q_j in which the term M_k take the form $M_k = M(p_k, \mathbf{P})$, and we can say in advance for exactly which separable metrics and potentials Q_j is a polynomial in the momenta. We give, in principle, a complete solution to this problem. Moreover, we show how to characterize each term M_k in (2) by the Poisson brackets $\{M_k, P_j\}$. [Note: Although the term $M_k(x_k, \mathbf{P})$ always exists, there are cases where it cannot be expressed as $M_k(p_k, \mathbf{P})$, i.e., as a function of p_k alone. These are exactly the cases where x_k is an *ignorable* variable, i.e., where the components of the metric tensor in the $\mathbf{x}$-coordinates do not depend on x_k and where, also, the potential V doesn't depend on x_k. However, these special cases where M_k, and the invariant Q_i of which it is a component term always have polynomial dependence (after multiplication by a linear combination of second-order invariants) can be handled separately or by requiring that M_k depends on a variable with some x dependence, such as $M_k(r(x_k)p_k, \mathbf{P})$ treated below.]

Of course, the system could admit a polynomial invariant

$$L = R(\mathbf{P}, Q_2, \cdots, Q_n)$$

such that $L, \mathbf{P}$ is functionally independent, even if $Q_2, \cdots, Q_n$ are not polynomials. It is a much more difficult problem to classify all such possibilities for polynomial L as functions of possibly nonpolynomial Q_j. We make some progress toward the solution of this problem, through the consideration of important examples. These questions of when a system with n second-order constants of the motion (generated by an orthogonal separation of variables) admits additional poynomial constants of the motion are closely related to the concept of superintegrability, [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16].

For dimension $n = 2$ in this paper, we go beyond the formulation discussed above and consider all cases where the Hamilton-Jacobi equation admits a 2nd-order constant of the motion, not necessarily associated with orthogonal separable coordinates, or even separable coordinates at all. In each of these cases we construct an additional constant of the motion.

2 Cartesian systems in two dimensions

Let us first consider two dimensional Euclidean space. In Cartesian coordinates the Hamiltonian H has the form

$$H = p_x^2 + p_y^2 + V(x, y).$$

If we have separation of variables in Cartesian coordinates the potential must take the form

$$V(x, y) = X(x) + Y(y). \tag{3}$$

We immediately observe that there are already two invariants arising from the separation, namely $L_1 = p_x^2 + X(x)$ and $L_2 = p_y^2 + Y(y)$. Our problem is to calculate a third invariant and determine when it can be chosen to be a polynomial in the canonical momenta. To do this we compute two functions $M(x, p_x)$ and $N(y, p_y)$ that satisfy the conditions

$$\{H, M\} = 1, \qquad \{H, N\} = 1. \tag{4}$$

These equations can be solved in principle if we know the original functions X and Y. Indeed if we write out the first of these conditions we obtain

$$2p_x \frac{\partial M}{\partial x} - X' \frac{\partial M}{\partial p_x} = 1.$$

This equation can be readily solved to give

$$M = -\int X'^{-1} dQ$$

where $Q = p_x$ and $L_1 = p_x^2 + X$. (We consider $X'^{-1} = \frac{dx}{dX}$ to be a function of $X = L_1 - Q^2$ to compute the integral. An arbitrary function $f(L_1, L_2)$ can be added to the integral, but this makes no difference since L_1, L_2 are invariants.) Once M and N have been determined, we see that $L_3 = N - M$ must be an invariant. It is immediately clear that if $X = x^{\frac{1}{p}}$ where p is an integer then M is a polynomial in p_x. As examples of this consider

1. $p = 3$.

$$M = -3x^{\frac{2}{3}} p_x - 4x^{\frac{1}{3}} p_x^3 - \frac{8}{5} p_x^5.$$

2. $p = 4$.

$$M = -4x^{\frac{3}{4}} p_x - 8x^{\frac{1}{2}} p_x^3 - \frac{32}{5} x^{\frac{1}{4}} p_x^5 - \frac{64}{35} p_x^7.$$

It follows from these two examples that the Hamiltonian

$$H = p_x^2 + p_y^2 + x^{\frac{1}{3}} + y^{\frac{1}{4}}$$

has in addition to the obvious invariants

$$L_1 = p_x^2 + x^{\frac{1}{3}}, \qquad L_2 = p_y^2 + y^{\frac{1}{4}},$$

the additional invariant

$$L_3 = 3x^{\frac{2}{3}} p_x + 4x^{\frac{1}{3}} p_x^3 + \frac{8}{5} p_x^5 - 4y^{\frac{3}{4}} p_y - 8y^{\frac{1}{2}} p_y^3 - \frac{32}{5} y^{\frac{1}{4}} p_y^5 - \frac{64}{35} p_y^7. \tag{5}$$

From this observation we conclude that all potentials of the form

$$V = \alpha x^{\frac{1}{p}} + \beta y^{\frac{1}{q}} \tag{6}$$

have the superintegrability property with three functionally independent invariants which are polynomial in p_x and p_y. This includes the known examples corresponding to $p = 1, 2$. If $X(x)$ is determined by a polynomial relation of the form

$$\sum_{j=1}^{n} a_j X^j = x$$

we can go even further. Then the function M is always a polynomial in the canonical momentum p_x. As an example consider

$$X(x) = 2^{-1/3}[\{x + \sqrt{x^2+1}\}^{1/3} - \{x + \sqrt{x^2+1}\}^{-1/3}]. \tag{7}$$

The inverse function is

$$x = X^3 + \frac{3}{2^{2/3}} X$$

and the corresponding function $M(x, p_x)$ is given by

$$-M(x, p_x) = \frac{8}{5} p_x^5 + 4X p_x^3 + 3X^2 p_x + \frac{3}{2^{2/3}} p_x.$$

It is clear that all that we have done applies also to potentials that separate in n dimensions, in Cartesian coordinates. There is only one further Cartesian case for which polynomial invariants can be generated. Let us consider the case when $X(x) = \omega_1^2 x^2$. The corresponding function $M(x, p_x)$ is given by

$$M(x, p_x) = \frac{1}{4\omega_1} \arcsin\left(\frac{\omega_1^2 x^2 - p_x^2}{\omega_1^2 x^2 + p_x^2}\right).$$

If $Y(y) = \omega_2^2 y^2$ this establishes that the Hamiltonian

$$H = p_x^2 + p_y^2 + \omega_1^2 x^2 + \omega_2^2 y^2 \tag{8}$$

has the constant of motion

$$L_3 = \frac{1}{4\omega_1} \arcsin\left(\frac{\omega_1^2 x^2 - p_x^2}{\omega_1^2 x^2 + p_x^2}\right) - \frac{1}{4\omega_2} \arcsin\left(\frac{\omega_2^2 y^2 - p_y^2}{\omega_2^2 y^2 + p_y^2}\right), \tag{9}$$

in addition to the constants $L_1 = p_x^2 + \omega_1^2 x^2$ and $L_2 = p_y^2 + \omega_2^2 y^2$. In general this invariant is not polynomial in the canonical momenta. However, if ω_1/ω_2 is a fraction p/q for integers p, q then $\omega_1 = ps, \omega_2 = qs$ and $L_3' = \sin(4spqL_3)$ will be a rational invariant whose common denominator is a product of powers of L_1

and L_2. The numerator is then an additional polynomial invariant, e.g., consider $\omega_1 = 1, \omega_2 = 2$. Then

$$L_3' = \sin(8L_3) = \frac{L_1 L_2^2 - 2(xp_y^2 - 4yp_xp_y - 4xy^2)^2}{L_1 L_2^2},$$

which indicates that $L_3'' = xp_y^2 - 4yp_xp_y - 4xy^2$ is an additional invariant. In general, $L_1^p L_2^q \sin(4spqL_3)$ will be a polynomial invariant, functionally independent of L_1 and L_2.

3 General two-dimensional separable systems

If we extend this problem to the case of orthogonal separable coordinates in a general Riemannian space, we know that the Hamiltonian in a given set of coordinates with a separable potential has the form

$$H = L_1 = \frac{p_x^2 + p_y^2 + v_1(x) + v_2(y)}{f_1(x) + f_2(y)}. \tag{10}$$

and, due to the separability, there is the invariant [17, 18]

$$L_2 = \frac{f_2(y)(p_x^2 + v_1(x)) - f_1(x)(p_y^2 + v_2(y))}{f_1(x) + f_2(y)}.$$

We can implement the same ansatz as we have done previously by looking for a function $M(H, x, p_x)$ which satisfies

$$\{H, M\} = \frac{1}{f_1(x) + f_2(y)}. \tag{11}$$

The condition has the form

$$(-v_1'(x) + f_1'(x)H)\frac{\partial M}{\partial p_x} + 2p_x\frac{\partial M}{\partial x} = 1. \tag{12}$$

Assuming that $|v_1'| + |f_1'| > 0$, we see that this equation has the solution

$$M(H, L_2, p_x) = \int U'^{-1} dQ$$

where

$$Q = p_x, \qquad L_2 = v_1(x) - f_1(x)H + p_x^2, \qquad U(x) = -v_1(x) + f_1(x)H + L_2.$$

(We consider $U'^{-1} = \frac{dx}{dU}$ to be a function of $U = Q^2$. An arbitrary function $f(L_1, L_2)$ can be added to the integral, but this makes no difference since

$L_1 = H$ and L_2 are invariants.) There is a similar condition for the function $N(H, L_2, y, p_y)$. The new invariant is $L_3 = N - M$. It is straightforward to verify the condition

$$\{L_2, L_3\} = 1. \tag{13}$$

Indeed, $\{L_2, M\} = f_2/(f_1 + f_2)$, $\{L_2, N\} = -f_1/(f_1 + f_2)$. This implies that the set L_1, L_2, L_3 is functionally independent.

Similarly we can construct functions $M(H, x, p_x)$, $N(H, y, p_y)$ that satisfy

$$\{H, M\} = \frac{f_1(x)}{f_1(x) + f_2(y)}, \qquad \{H, N\} = \frac{-f_2(y)}{f_1(x) + f_2(y)}, \tag{14}$$

Assuming that $|v_i'| + |f_i'| > 0$ for $i = 1, 2$, we see that these equations have the solutions

$$M(H, L_2, p_x) = \int f_1(x) {U_1'}^{-1} dQ, \qquad N(H, L_2, p_y) = -\int f_2(y) {U_2'}^{-1} dQ$$

where

$$U_i = -v_i + f_i H + L_2.$$

Setting $L_4 = N - M$, we see that L_4, not an invariant, satisfies

$$\{H, L_4\} = 1, \qquad \{L_2, L_4\} = 0. \tag{15}$$

Let us illustrate what can happen with some examples.

1. We choose parabolic coordinates in Euclidean space $x' = \frac{1}{2}(\xi^2 - \eta^2)$, $y' = \xi\eta$. First consider the parabolic-separable Hamiltonian

$$H = L_1 = \frac{p_\xi^2 + p_\eta^2 + \xi}{\xi^2 + \eta^2}. \tag{16}$$

We can immediately associate with this the extra invariant

$$L_2 = \frac{\eta^2 p_\xi^2 - \xi^2 p_\eta^2 + \eta^2 \xi}{\xi^2 + \eta^2}.$$

If we look for our functions $M(\xi, p_\xi)$ and $N(\eta, p_\eta)$, as before we obtain

$$M(\xi, p_\xi) = \frac{1}{4\sqrt{H}} \ln\left(\frac{\sqrt{H} p_\xi + \frac{1}{2} - \xi H}{-\sqrt{H} p_\xi + \frac{1}{2} - \xi H}\right),$$

$$N(\eta, p_\eta) = \frac{1}{4\sqrt{H}} \ln\left(\frac{\sqrt{H}\eta + p_\eta}{\sqrt{H}\eta - p_\eta}\right).$$

If we now consider the constant $\cosh(4(M-N)\sqrt{H})$, we find that it can be written in the form

$$4\cosh(4(M-N)\sqrt{H}) = \frac{L_3^2 H}{(1-4HL_2)L_2},$$

where

$$L_3 = \frac{2\xi\eta}{\xi^2+\eta^2}(p_\xi^2+p_\eta^2) - 2p_\xi p_\eta + \frac{\eta(\xi^2-\eta^2)}{\xi^2+\eta^2} \qquad (17)$$

is an additional invariant quadratic in the canonical momenta [19].

2. Consider the Hamiltonian in Cartesian coordinates

$$H = p_x^2 + p_y^2 + \frac{x}{\sqrt{x^2+y^2}}. \qquad (18)$$

In parabolic coordinates this Hamiltonian has the form

$$H = L_1 = \frac{p_\xi^2 + p_\eta^2 + \xi^2 - \eta^2}{\xi^2+\eta^2}.$$

The second order invariant associated with this separation is

$$L_2 = \frac{\xi^2 p_\eta^2 - \eta^2 p_\xi^2 - 2\xi^2\eta^2}{\xi^2+\eta^2}.$$

The additional invariant calculated by our method is given by

$$L_3 = \frac{\text{arccosh}(\frac{(H-1)\xi^2+p_\xi^2}{(H-1)\xi^2-p_\xi^2})}{\sqrt{H-1}} + \frac{\text{arccosh}(\frac{(H+1)\eta^2+p_\eta^2}{(H+1)\eta^2-p_\eta^2})}{\sqrt{H+1}}, \qquad (19)$$

which is clearly transcendental.

3. If we consider the Hamiltonian

$$H = p_x^2 + p_y^2 + ib(x+iy), \qquad (20)$$

then using the semihyperbolic coordinates

$$x+iy = i(u+w), \qquad x-iy = (-i/2)(u-w)^2$$

and applying our construction, we find

$$\frac{\exp(M-N)-i}{\exp(M-N)+i} = -i\frac{\sqrt{b-iX}}{\sqrt{b+iX}},$$

thus giving rise to the additional constant $X = p_x + ip_y$.

4. Let's now look at an example of a potential where our construction yields elliptic integrals. We consider the potential $V = 2x + \frac{\beta}{y^2}$. If we carry out the construction using parabolic coordinates $x = (1/2)(\xi^2 - \eta^2)$, $y = \xi\eta$ then the functions M and N are given by the integrals

$$M = \frac{1}{2}\int \frac{\xi d\xi}{\sqrt{-\xi^6 + H\xi^4 + L\xi^2 - \beta}}, \quad N = \frac{1}{2}\int \frac{\eta d\eta}{\sqrt{\eta^6 + H\eta^4 + L\eta^2 - \beta}},$$

where L is the quadratic constant associated with the separation of variables in parabolic coordinates. If we change variables according to $u = \xi^2$, $v = -\eta^2$ then both M and N are given by integrals of the form

$$I = \frac{1}{2}\int \frac{d\lambda}{\sqrt{(a-\lambda)(b-\lambda)(c-\lambda)}},$$

where $\lambda = u, v$ and

$$abc = -\beta, \qquad L = ab + bc + ac, \qquad H = a + b + c.$$

There are a variety of ways of evaluating elliptic integrals of this type. We recall that all our considerations are in the complex domain. As an example, we can choose to use the complex equivalent of the integral

$$\int_{-\infty}^{u} \frac{dx}{\sqrt{(a-x)(b-x)(c-x)}} = \frac{2}{\sqrt{a-c}} F(\alpha, p),$$

valid for $a > b > c \geq u$ and for which

$$\sqrt{\frac{a-c}{a-u}} = \sin\alpha = \mathrm{sn}(A, p), \qquad p = \sqrt{\frac{a-b}{a-c}}.$$

Then if we calculate $\mathrm{sn}^2(\sqrt{a-c}(M-N), p)$ using the addition formulas for elliptic functions we obtain

$$\mathrm{sn}^2(\sqrt{a-c}(M-N), p) = \frac{c-a}{c+b+L_1},$$

where L_1 is the second quadratic constant associated with this superintegrable system. Because of the various ways of evaluating elliptic integrals there are a number of ways of uncovering the presence of L_1.

In analogy with the constructions (5)-(7), we can find Riemannian spaces and potentials with polynomial invariants of arbitrarily high order. Set

$$x = P_n\left(\frac{U+A}{\alpha+\beta H}\right), \qquad A = \delta + \phi H - L_2, \tag{21}$$

where P_n is a polynomial of order n and $\alpha, \beta, \delta, \phi$ are constants. Then there exists a function F_n, inverse to P_n, i.e., $F_n(P_n(y)) = y$, such that

$$U = (\alpha + \beta H)F_n(x) - \delta - \phi H + L_2,$$

and $v_1(x) = -\alpha F_n(x) + \delta$, $f_1(x) = \beta F_n(x) - \phi$, where $(\alpha + \beta H)^n M(x, p_x)$ is a polynomial in the momenta. The cartesian coordinate constructions (5)-(7) correspond to the special case $\beta = 0$.

The solution of the equation (11) can be understood in a more general context. We have the dual relations

$$x = F(U - L_2, H), \qquad U(x, H) = -v_1(x) + f_1(x)H + L_2, \quad U_x \neq 0. \quad (22)$$

(Since U and L_2 occur only as $U - L_2$ we will, without loss of generality, set $L_2 = 0$ in the theoretical developments to follow, and then replace U by $U - L_2$ in the examples.) Thus we have

$$1 = F_U U_x, \qquad F_U U_H + F_H = 0.$$

The condition that $U(x, H)$ is linear in H, i.e., $U_{HH} = 0$, leads to the following necessary and sufficient conditions that the function $x = F(U, H)$ correspond to an invariant M on a Riemannian manifold with potential:

$$F_{HH}F_U^2 - 2F_{UH}F_U F_H + F_{UU}F_H^2 = 0, \quad F_U \neq 0. \quad (23)$$

This equation admits an infinite dimensional conformal symmetry group. Indeed if $V = F(U, H)$ is a solution then $G(V)$ is also a solution, for *any* nonconstant function G. Also, this group contains the subgroup of inhomogeneous affine symmetries: if $F(U, H)$ is a solution then so is $F([a_{11}U + a_{12}H + +a_{13}]/A, [a_{21}U + a_{22}H + a_{23} + a_{23}]/A)$, where a_{ij} are constants, $\det(a_{ij}) \neq 0$ and

$$A = a_{31}U + a_{32}H + +a_{33}.$$

Note that the function $V_1 = (U + \delta + \phi H)/(\alpha + \beta H)$ satisfies (23), so any function of V_1 must also satisfy the requirement. This puts (21) in the proper context. A more general solution is $V_2 = (U + \phi H + \delta)/(\alpha U + \beta H + \gamma)$, where again any function of V_2 also satisfies the requirement. Equation (23) also occurs in the theory of level sets, used in computational geometry and computer vision, [20], since it describes the family of functions F whose level sets are always straight lines in the (U, H) plane.

We have seen that the construction (21) always leads to a polynomial invariant L_3, up to multiplication by a polynomial in H and L_2. In fact these are the *only* polynomial invariants L_3 that can be constructed directly from the integration. This follows from

Theorem 1 *The function $F(U,H)$ with $F_U \neq 0$ is a solution of equation (21) with polynomial dependence on U if and only if it is of the form*

$$F(U,H) = P\left(\frac{U+\alpha H+\beta}{\gamma H+\delta}\right)$$

where P is a (nonconstant) polynomial and $\alpha, \beta, \gamma, \delta$ are constants with $|\gamma|^2 + |\delta|^2 > 0$.

PROOF: Let

$$F = a_0(H)U^N + a_1(H)U^{N-1} + \cdots + a_{N-1}(H)U + a_N(H)$$

be a solution of (21) with $N \geq 1$ and $a_0 \neq 0$. Substituting this expression into (21) and equating the coefficient of U^{3N-2} on both sides of the resulting expression, we find the condition $a_0''a_0 = (\frac{N+1}{N})a_0'^2$, so $a_0(H) = (\gamma H+\delta)^{-N}$. Now we make the change of variables $\tilde{U} = \frac{U}{\gamma H+\delta}, \quad \tilde{H} = \frac{\phi H+\rho}{\gamma H+\delta}$, where $\phi\delta - \gamma\rho \neq 0$. It follows that

$$F = \tilde{U}^N + \tilde{a}_1(\tilde{H})\tilde{U}^{N-1} + \cdots + \tilde{a}_{N-1}(\tilde{H})\tilde{U} + \tilde{a}_N(\tilde{H})$$

in the new coordinates, and F is a solution of

$$F_{\tilde{H}\tilde{H}}F_{\tilde{U}}^2 - 2F_{\tilde{U}\tilde{H}}F_{\tilde{U}}F_{\tilde{H}} + F_{\tilde{U}\tilde{U}}F_{\tilde{H}}^2 = 0. \tag{24}$$

Substituting the polynomial into (24) and equating coefficients of $\tilde{U}^{3N-3}$, we find $\tilde{a}_1''N^2 = 0$ or $\tilde{a}_1 = \alpha_1\tilde{H} + \beta_1$. Using this information, we return to our original expression for the polynomial and make a new change of variables of the form

$$\tilde{U} = \frac{U+\alpha H+\beta}{\gamma H+\delta}, \qquad \tilde{H} = \frac{\chi H+\zeta}{\gamma H+\delta}, \tag{25}$$

where $\chi\delta - \gamma\zeta \neq 0$, and α, β are chosen such that the transformed coefficient of $\tilde{U}^{N-1}$ vanishes. In these variables

$$F = \tilde{U}^N + \tilde{a}_2(\tilde{H})\tilde{U}^{N-2} + \cdots + \tilde{a}_{N-1}(\tilde{H})\tilde{U} + \tilde{a}_N(\tilde{H}).$$

We substitute this expression into (24), and equating coefficients of $\tilde{U}^{3N-4}$ we find $\tilde{a}_2'' = 0$, so $\tilde{a}_2$ is a polynomial in $\tilde{H}$ of order ≤ 1. Proceeding in this fashion to equate coefficients of $\tilde{U}^{3N-s}$ for $s = 5, 6, \cdots$ in order, we find that the first occurence of $\tilde{a}_k, k \geq 3$ in this sequence of equations takes the form $\tilde{a}_k'' = p_k(\tilde{a}_2, \cdots, \tilde{a}_{k-1})$ where p_k is a polynomial of order 3 at most. It follows by induction on k that each $\tilde{a}_k$ is a polynomial in $\tilde{H}$.

At his point we have shown that F is a polynomial in both $\tilde{U}$ and in $\tilde{H}$. Let $\tilde{H}^M$ be the maximal power of $\tilde{H}$ that occurs in F. If $M = 0$ we are done. Assume $M \geq 1$. If we use the argument of the first paragraph of this proof with $\tilde{U}$

and $\tilde{H}$ interchanged, we see that the coefficient of $\tilde{H}^M$ in F must take the form $\alpha_0/(\beta_1\tilde{U}+1)$ with $\alpha_0 \neq 0$. Since F is a polynomial in $\tilde{U}$ we must have $\beta_1 = 0$.

Thus

$$F = \tilde{U}^N + \tilde{a}_2(\tilde{H})\tilde{U}^{N-2} + \cdots + \tilde{a}_{N-1}(\tilde{H})\tilde{U} + \alpha_0\tilde{H}^M.$$

Now substitute this expression into (24) and equate coefficients of $\tilde{U}^n\tilde{H}^m$ where $n+m$ is maximal. Suppose $N \geq M$. The highest power term in $F_{\tilde{H}\tilde{H}}F_{\tilde{U}}^2$ is $\alpha_N M(M-1)N^2\tilde{H}^{M-2}\tilde{U}^{2N-2}$. The highest power term in $F_{\tilde{U}\tilde{U}}F_{\tilde{H}}^2$ is $\alpha_N^2 N(N-1)M^2\tilde{H}^{2M-2}\tilde{U}^{N-2}$, but this is of lower order. The highest power term in $2F_{\tilde{U}\tilde{H}}F_{\tilde{H}}F_{\tilde{U}}$ is $t = 2\alpha_N a_{N_1,M_1}N_1M_1NM\tilde{U}^{N_1+N-2}\tilde{H}^{M_1+M-2}$ where a_{N_1,M_1} is the coefficient of $\tilde{U}^{N_1}\tilde{H}^{M_1}$ in F. Here $N_1 < N, M_1 < M$. If $N > N_1 + M_1$ then the highest power term is the coefficient of $\tilde{H}^{M-2}\tilde{U}^{2N-2}$, so $M = 1$. If $N \leq N_1 + M_1$ then $t = 0$, so $a_{N_1,M_1} = 0$. Thus, the only possiblity is $M = 1$, so

$$F = \tilde{U}^N + \alpha_2\tilde{U}^{N-2} + \cdots + \alpha_{N-1}\tilde{U} + \alpha_N\tilde{H}.$$

Substituting this expression into the differential equation we see that $F_{\tilde{U}\tilde{U}} = 0$, or $F = \tilde{U} + \alpha_N\tilde{H}$. But this is impossible since $N = 1$ and the coefficient of $\tilde{U}^{N-1}$ must be 0. Hence F depends only on $\tilde{U}$. There is a similar argument for the case $M > N$. **QED**

If we limit our search for potentials to a space in which $U_H = f_1(x)$ is prescribed, then the general conditions (23) are replaced by

$$F_H + f_1(F)F_U = 0, \quad F_U \neq 0. \tag{26}$$

Equation (26) admits the complete integral

$$F(U,H,\alpha,\beta) = f_1^{-1}\left(\frac{U+\alpha}{H+\beta}\right),$$

where f_1^{-1} is the function inverse to f_1. From this one can use standard techniques (method of characteristics, envelopes of solutions) from the theory of quasilinear first order partial differential equations to construct solutions of (26) that satisfy particular initial conditions or that depend on arbitrary functions, [21] (chapter II) or [22] (section 88).

Note: Standard Hamilton-Jacobi theory gives essentially these same constants of the motion, but from a different viewpoint, [1]. Our expression for L_3, for example, is

$$L_3 = \int U_x'^{-1}dp_x - \int U_y'^{-1}dp_y = M - N,$$

where $U_x = -v_1(x) + f_1(x)H + L_2$, etc. Standard Hamilton-Jacobi theory gives

$$L_3 = \frac{1}{2}\int \frac{dx}{\sqrt{-v_1 + f_1H + L_2}} - \frac{1}{2}\int \frac{dy}{\sqrt{-v_2 + f_2H - L_2}} = \tilde{M} - \tilde{N}.$$

In the standard theory $\tilde{M} = \tilde{M}(H, L_2, x)$, etc., whereas in our approach $M = M(H, L_2, p_x)$, etc. In both cases the condition (12) is satisfied. Our approach makes it easier in some cases to determine if polynomial invariants exist. It also points out the bracket relations between M, N and the operators L_j defining the separation, e.g., (11).

Examples abound of spaces for which these constructions apply. We illustrate this with a family of surfaces in Minkowski space: $ds^2 = dz^2 - dy^2 - dx^2$. The surfaces involve a horispherical coordinate ξ and take the form

$$\mathbf{X}(t, \xi) = (x, y, z) = \Big(2t\xi, g(t) + (\xi^2 - 1)t, g(t) + (\xi^2 + 1)t\Big) . \qquad (27)$$

The metric on the surface is

$$ds^2 = 4[tg'(t)\, dt^2 - t^2\, d\xi^2] = 4t^2[d\rho^2 - d\xi^2] = (f(\rho) + 1)[d\rho^2 - d\xi^2],$$

where $(\frac{d\rho}{dt})^2 = \frac{g'(t)}{t^2}$, and we can construct a polynomial invariant for the surface (and for an appropriate added potential) provided that the function $t^2 = F(\rho)$ has a polynomial inverse function, i.e., $\rho = G(t^2)$ where G is a polynomial. Clearly $g'(t) = 4t^4 G'(t^2)^2$ and any polynomial G will determine a surface with a polynomial invariant. For example, choose $G(t^2) = \frac{1}{2}t^4 + t^2$. Then we can take $g(t) = \frac{4}{9}t^9 + \frac{8}{7}t^7 + \frac{4}{5}t^5$ and $\rho(t) = \frac{1}{2}t^4 + t^2$. The resulting M will be third-order polynomial in p_ξ and p_ρ. Similarly, we can determine a potential term $v(\rho)$ with $v' \neq 0$ such that N is a polynomial in p_ξ and p_ρ.

Rather than make either of the choices p_x or x for the independent variable in (12) we could choose some other function $w(x, p_x)$, adapted to the specific problem at hand. For example, let us take $w(x, p_x) = r'(x)p_x$ for some given function r, and require $M = M(H, L_2, w)$. Solving (12) in these variables we find

$$M = \int \frac{dr(x)}{dw^2}\, dw, \qquad (28)$$

where

$$w^2 = U = r'(x)^2 p_x^2 = r'(x)^2(-v_1 + f_1 H + L_2), \qquad r(x) = F(U, H, L_2).$$

This approach will work even if v_1 and f_1 are constants; it is guaranteed to yield a polynomial invariant if we require

$$r = P_n\left(\frac{U + \alpha_1 H + \alpha_2 L_2 + \alpha_3}{\alpha_4 H + \alpha_5 L_2 + \alpha_6}\right), \qquad (29)$$

where P_n is a polynomial of order n and the α_i are constants. Then there exists a function F_n, inverse to P_n, such that

$$U = (\alpha_4 H + \alpha_5 L_2 + \alpha_6)F_n(r) - (\alpha_1 H + \alpha_2 L_2 + \alpha_3) = r'^2(-v_1 + f_1 H + L_2).$$

Equating coefficients of L_2 we find the condition $r'(x)^2 = \alpha_5 F_n(r) - \alpha_2$ and we can solve for $r(x)$ by quadratures. Equating coefficients of H and the constant term, we obtain expressions for f_1 and v_1:

$$f_1(x) = \frac{\alpha_4 F_n(r) - \alpha_1}{\alpha_5 F_n(r) - \alpha_2}, \qquad v_1(x) = \frac{\alpha_3 - \alpha_6 F_n(r)}{\alpha_5 F_n(r) - \alpha_2}.$$

It follows that $(\alpha_4 H + \alpha_5 L_2 + \alpha_6)^n M(rp_x)$ is a polynomial in the momenta.

4 Lie form and nonorthogonal separation in two dimensions

We know that if a Hamiltonian

$$H = \sum_{i,j=1}^{2} g^{ij} p_i p_j$$

admits a constant of the motion L that is quadratic in the momenta

$$L = \sum_{i,j=1}^{2} a^{ij} p_i p_j, \qquad \{H, L\} = 0 \tag{30}$$

and if the roots of the determinant $|a^{ij} - \lambda g^{ij}|$ are distinct, then the eigenforms define new (separable) variables ρ, μ and the Hamiltonian can be written in Liouville form

$$H = \frac{p_\rho^2 + p_\mu^2}{f(\rho) + g(\mu)}.$$

However, it may be that the roots of this determinant are equal. In this case H cannot be put into Liouville form, but rather Lie form, which for a suitable choice of variables (non-separable) is

$$H = \frac{p_x p_y}{x + B(y)}. \tag{31}$$

The associated quadratic constant of the motion is

$$L = p_x^2 - 2yH. \tag{32}$$

We now ask the question: When the roots of L are equal, how can we calculate the third invariant? We are interested in the the same question when a potential is added to the Hamiltonian. These questions can readily be answered. Indeed if we look for a function $N(H, L, y, p_y)$ that is in involution with H, we obtain the equation

$$(x + B(y))N_y + p_y B'(y)N_{p_y} = 0. \tag{33}$$

If we solve (31), (32) for x and p_x in terms of the variables H, L, y and p_y, we obtain

$$p_x = \sqrt{L + 2yH}, \qquad x = \frac{p_y}{H}\sqrt{L + 2yH} - B(y).$$

The equation (33) for N then has the form

$$\frac{\sqrt{L + 2yH}}{HB'(y)} N_y + N_{p_y} = 0.$$

From this condition a second invariant can be readily obtained in the form

$$L' = H \int \frac{B'(y)}{\sqrt{L + 2yH}} dy - p_y. \tag{34}$$

We now extend these considerations by considering the possibility of adding a potential. If we do this and have an extra quadratic constant then H and L have the forms

$$H = \frac{p_x p_y + \frac{1}{2}K(y)}{x + B(y)} + \frac{1}{2}U'(y), \quad L = p_x^2 - 2yH + U(y). \tag{35}$$

Solving (35) for p_x and x gives

$$p_x = \sqrt{L - U(y) + 2yH}, \qquad x = \frac{p_y\sqrt{L - U(y) + 2yH} + \frac{1}{2}K(y)}{H - \frac{1}{2}U'(y)} - B(y).$$

Then the equation for N has the form

$$2\sqrt{L - U(y) + 2yH}(2H - U'(y))N_y \tag{36}$$

$$+[-2U''(y)\sqrt{L - U(y) + 2yH}p_y + B'(y)U'(y)^2 + 4B'(y)H^2 - U''(y)K(y)$$
$$-4B'(y)U'(y)H + K'(y)U'(y) - 2K'(y)H]N_{p_y} = 0.$$

This equation can, in principle, be solved directly. In fact for suitable redefinition of the variables $y \to Y$, $p_y \to P_y$ equation (36) can be put in the form

$$N_Y + (P_Y + s(Y))N_{P_Y} = 0 \tag{37}$$

that can be solved by the further transformation

$$P_{Y'} = P_Y + t(Y), \qquad Y' = Y.$$

Then, provided that

$$t'(Y) - t(Y) + s(Y) = 0,$$

(37) reduces to

$$N_{Y'} + P_{Y'} N_{P_{Y'}} = 0.$$

From this we immediately deduce an extra constant of the motion of the form

$$L' = e^{Y'}/P_{Y'}. \tag{38}$$

The equation for $t(Y)$ has the solution

$$t(Y) = e^Y \int^Y e^{-u} s(u) du.$$

There is one remaining possibility for a quadratic constant of the motion (30) in two dimensions: the constant may be associated with *nonorthogonal* separation of variables. In two dimensions there is only one case: separation in light cone (null) coordinates, []. For this case the Hamiltonian takes the form

$$H = p_z p_{\bar{z}} + f(\bar{z})$$

and there is a Killing vector p_z, so p_z^2 is a second-order constant of the motion. In addition there is a quadratic constant

$$L = M p_z + \frac{i}{2} \int \bar{z} \frac{df}{d\bar{z}} d\bar{z}.$$

Thus we have answered the following questions.

1. If a Hamiltonian with potential admits a quadratic constant of the motion in two dimensions how does one calculate the third constant?

2. A subset of problem 1 is when we require separation only and ask to calculate the third constant.

5 Systems in three dimensions

Let us now look at how the orthogonal separation of variable considerations extend to three dimensions. If we have a general separable coordinate system in three dimensions we could take the Hamiltonian to be [18, 23]

$$\begin{aligned} H = L_1 &= \frac{g_2 - g_3}{\Phi}(p_{x_1}^2 + v_1(x_1)) + \frac{g_3 - g_1}{\Phi}(p_{x_2}^2 + v_2(x_2)) \\ &+ \frac{g_1 - g_2}{\Phi}(p_{x_3}^2 + v_3(x_3)) \end{aligned} \tag{39}$$

where $g_i = g_i(x_i), f_i = f(x_i)$ and Φ is the determinant of the Stäckel matrix

$$\begin{pmatrix} 1 & f_1 & g_1 \\ 1 & f_2 & g_2 \\ 1 & f_3 & g_3 \end{pmatrix} \tag{40}$$

This automatically gives us two more invariants:

$$\begin{aligned} L_2 &= \frac{f_3 - f_2}{\Phi}(p_{x_1}^2 + v_1(x_1)) + \frac{f_1 - f_3}{\Phi}(p_{x_2}^2 + v_2(x_2)) \\ &+ \frac{f_2 - f_1}{\Phi}(p_{x_3}^2 + v_3(x_3)), \end{aligned} \tag{41}$$

$$\begin{aligned} L_3 &= \frac{f_2 g_3 - f_3 g_2}{\Phi}(p_{x_1}^2 + v_1(x_1)) + \frac{f_3 g_1 - f_1 g_3}{\Phi}(p_{x_2}^2 + v_2(x_2)) \\ &+ \frac{f_1 g_2 - f_2 g_1}{\Phi}(p_{x_3}^2 + v_3(x_3)). \end{aligned} \tag{42}$$

We need to find an additional two invariants, such that the five form a functionally independent set.

If we look for a function M_1 such that

$$\{H, M_1\} = \frac{g_2 - g_3}{\Phi}, \tag{43}$$

then this function satisfies the equation

$$2p_{x_1}\partial_{x_1}M_1 + [-v_1'(x_1) + f_1'H + g_1'L_2]\partial_{p_{x_1}}M_1 = 1, \tag{44}$$

which looks like the form we have been using in two dimensions. There are similar equations for the corresponding functions M_i for $i = 2, 3$. For $M_1(H, L_2, L_3, Q_1)$ with $Q_1 = p_{x_1}$ this has the solution

$$M_1 = \int U_1'^{-1} dQ_1$$

where $U_1(x_1) = -v_1(x_1) + f_1H + g_1L_2 + L_3$ and $L_3 = v_1 - f_1H - g_1L_2 + p_{x_1}^2$. (Here, we consider $U_1'^{-1} = \frac{dx_1}{dU_1}$ to be a function of $U_1 = Q_1^2$ to compute the integral. We also assume that $|v_1'| + |f_1'| + |g_1'| > 0$.) The corresponding invariant that we can calculate from these three functions is $L_3' = M_1 + M_2 + M_3$. This is based on the obvious identity

$$(g_2 - g_3) + (g_3 - g_1) + (g_1 - g_2) = 0.$$

Note: As in the two-dimensional case, the solution of the equation (44) can be understood in a more general context. We have the dual relations

$$x = F(U - L_3, H, L_2), \quad U(x, H, L_2) = -v_1(x) + f_1(x)H + g_1(x)L_2 + L_3, \tag{45}$$

where $U_x \neq 0$. (Since U and L_3 occur only as $U - L_3$ we can, without loss of generality, set $L_3 = 0$ in the equations immediately following, and then replace U by $U - L_3$ in the examples.) Thus we have

$$1 = F_U U_x, \qquad F_U U_H + F_H = 0, \qquad F_U U_{L_2} + F_{L_2} = 0.$$

The condition that $U(x, H, L_2)$ is linear in H and L_2, i.e., $U_{HH} = U_{L_2 L_2} = U_{HL_2} = 0$, leads to the following necessary and sufficient conditions that the function $x = F(U, H, L_2)$ correspond to an invariant M_1 on a Riemannian manifold with potential:

$$\begin{aligned} F_{HH}F_U^2 - 2F_{UH}F_UF_H + F_{UU}F_H^2 &= 0, \quad F_U \neq 0, \\ F_{UU}F_{L_2}^2 - 2F_{L_2U}F_{L_2}F_U + F_{L_2L_2}F_U^2 &= 0, \\ F_{L_2L_2}F_H^2 - 2F_{HL_2}F_HF_{L_2} + F_{HH}F_{L_2}^2 &= 0 \end{aligned} \tag{46}$$

These equations admit an infinite dimensional conformal symmetry group. Indeed if $V = F(U, H, L_2)$ is a solution then $G(V)$ is also a solution, for *any* nonconstant function G. Also, this group contains the subgroup of inhomogeneous affine symmetries: if $F(U, H, L_2)$ is a solution then so is $F([a_{11}U + a_{12}H + a_{13}L_2 + a_{14}]/A, [a_{21}U + a_{22}H + a_{23}L_2 + a_{24}]/A, [a_{31}U + a_{32}H + a_{33}L_2 + a_{24}]/A)$ where a_{ij} are constants, $\det(a_{ij}) \neq 0$ and

$$A = a_{41}U + a_{42}H + a_{43}L_2 + a_{44}.$$

As in the two-dimensional case, the only polynomial functions F of U are of a very special form.

Theorem 2 *The function $F(U, H, L_2)$ with $F_U \neq 0$ is a solution of equations (46) with polynomial dependence on U if and only if it is of the form*

$$F(U, H, L_2) = P\left(\frac{U + \alpha_1 H + \alpha_2 L_2 + \beta}{\gamma_1 H + \gamma_2 L_2 + \delta}\right)$$

where P is a (nonconstant) polynomial and $\alpha_i, \beta, \gamma_i, \delta$ are constants with $|\gamma_1|^2 + |\gamma_2|^2 + |\delta|^2 > 0$.

PROOF: The proof is similar to that of Theorem 1. It follows from this theorem and the first two equations (46) that

$$F = P^{(1)}(U^{(1)}, L_2) = P^{(2)}(U^{(2)}, H)$$

where, the $P^{(i)}$ are polynomials of strict order N in their first arguments and

$$U^{(1)} = \frac{U + \alpha_1^{(1)} L_2 H + \beta^{(1)} L_2}{\gamma_1^{(1)} L_2 H + \delta^{(1)} L_2} \quad U^{(2)} = \frac{U + \alpha_1^{(2)} H L_2 + \beta^{(2)} H}{\gamma_1^{(2)} H L_2 + \delta^{(2)} H}.$$

Furthermore the coefficients of the $N-1$-st power of their first arguments can be asumed to be zero. Comparing the coefficients of the highest power U^N of U in F, we see that this coefficient must be of the form

$$(\gamma_1 H + \gamma_2 L_2 + \gamma_3 H L_2 + \delta)^{-N},$$

where now the γ_i, δ are constants. Substituting this into the third equation (46) and equating coefficients of U^{3N}, we see that $\gamma_3 = 0$.

Equating the coefficients of U^{N-1} in the $P^{(i)}$ we see that

$$U^{(1)} = U^{(2)} = \tilde{U} = \frac{U + \alpha_1 H + \alpha_2 L_2 + \phi H L_2 + \beta}{\gamma_1 H + \gamma_2 L_2 + \delta}$$

where the coefficients are constants. Then, substituting this result into the 3rd equation again and comparing coefficients of U^{3N-1} we see that $\phi = 0$. At this point we have shown that $F = P(\tilde{U}, H, L_2)$ where P is a polynomial of order exactly N in its first argument. The proof that P is independent of its second and third arguments follows exactly as in the last part of the proof of Theorem 1. **QED**

If we limit our search for potentials to a space in which $U_H = f_1(x), U_{L_2} = g_1(x)$ are prescribed, then the general conditions (46) are replaced by

$$F_H + f_1(F) F_U = 0, \quad F_{L_2} + g_1(F) F_U = 0, \quad F_U \neq 0. \tag{47}$$

From this one can use standard techniques (method of characteristics, envelopes of solutions) from the theory of systems of quasilinear first order partial differential equations to construct solutions of (47) that satisfy particular initial conditions or that depend on arbitrary functions.

The invariant $L_3' = M_1 + M_2 + M_3$ also commutes with L_2. Indeed, from the fact that

$$\partial_{x_1} L_2 = \frac{f_3 - f_2}{\Phi} \left(v'_1 - f_1' H - g_1' L_2 \right)$$

we can verify that (44) implies

$$\{L_2, M_1\} = \frac{f_3 - f_2}{\Phi}, . \tag{48}$$

The corresponding conditions are satisfied by M_2 and M_3. Then the fact that $\{L_2, L_3'\} = 0$ is implied by the obvious identity

$$(f_3 - f_2) + (f_1 - f_3) + (f_2 - f_1) = 0.$$

Finally, from the fact that

$$\partial_{x_1} L_3 = \frac{f_2 g_3 - f_3 g_2}{\Phi} \left(v'_1 - f_1' H - g_1' L_2 \right)$$

we can verify that (44) implies

$$\{L_3, M_1\} = \frac{f_2 g_3 - f_3 g_2}{\Phi}, . \tag{49}$$

The corresponding conditions are satisfied by M_2 and M_3. Then the fact that $\{L_3, L_3'\} = 1$ is implied by the identity

$$(f_2 g_3 - f_3 g_2) + (f_3 g_1 - f_1 g_3) + (f_1 g_2 - f_2 g_1) = \Phi. \tag{50}$$

Similarly, we can define a new invariant L_2' by requiring that a new function M_1 satisfy

$$\{L_1, M_1\} = \frac{g_1 (g_2 - g_3)}{\Phi}, \tag{51}$$

with analogous conditions for M_2 and M_3. For $M_1(H, L_2, L_3, Q_1)$ with $Q_1 = p_{x_1}$ this has the solution

$$M_1 = \int g_1 U_1'^{-1} dQ_1$$

where $U_1(x_1) = -v_1(x_1) + f_1 H + g_1 L_2 + L_3$.

[Note that for M_1 to be a polynomial in p_x, p_y, p_z we must have $g_1(F) F_U$ a polynomial in U. If $g_1' = 0$ this reduces to requiring F to be a polynomial in U. If $g_1' \neq 0$ we can replace the variable x by $\tilde{x}_1 = r(x_1) = \int g_1(x_1)\, dx_1$ with $\tilde{x}_1 = G(U, H, L_2, L_3)$. Then $g_1(F) F_U = G_U$ and our original analysis goes through with F replaced by G. It is guaranteed to yield a polynomial invariant if we require

$$r = P_n \left(\frac{U + \alpha_1 H + \alpha_2 L_2 + \alpha_3 L_3 + \alpha_4}{\alpha_5 H + \alpha_6 L_2 + \alpha_7} \right), \qquad g_1 = r'(x_1) \tag{52}$$

where P_n is a polynomial of order n and the α_i are constants. Then there exists a function F_n, inverse to P_n, such that

$$U = (\alpha_5 H + \alpha_6 L_2 + \alpha_7) F_n(r) - (\alpha_1 H + \alpha_2 L_2 + \alpha_3 L_3 + \alpha_4) = -v_1 + f_1 H + g_1 L_2 + L_3.$$

Equating coefficients of L_2 we find the condition $r' = \alpha_6 F_n(r) - \alpha_2$ and we can solve for $r(x_1)$ by quadratures. Equating coefficients of H, L_3 and the constant term, we find $\alpha_3 = -1$ and

$$f_1(x) = \alpha_5 F_n(r) - \alpha_1, \quad g_1(x) = \alpha_6 F_n(r) - \alpha_2, \quad v_1(x) = \alpha_4 - \alpha_7 F_n(r).$$

It follows that $(\alpha_5 H + \alpha_6 L_2 + \alpha_7)^n M_1$ is a polynomial in the momenta.]

The corresponding invariant that we can calculate from these three functions is $L_2' = M_1 + M_2 + M_3$. This is based on the obvious identity

$$g_1(g_2 - g_3) + g_2(g_3 - g_1) + g_3(g_1 - g_2) = 0.$$

Then it follows that

$$\{L_2, M_1\} = \frac{g_1(f_3 - f_2)}{\Phi}, \qquad \{L_3, M_1\} = \frac{g_1(f_2 g_3 - f_3 g_2)}{\Phi},$$

with analogous results for M_2, M_3. Thus, from the definition of Φ we see that $\{L_2, L_2'\} = 1$.

Finally we define a function $L_1' = M_1 + M_2 + M_3$ by requiring

$$\{L_1, M_1\} = \frac{f_1(g_2 - g_3)}{\Phi}, \tag{53}$$

with similar conditions for M_2 and M_3. For $M_1(H, L_2, L_3, Q_1)$ with $Q_1 = p_{x_1}$ this has the solution

$$M_1 = \int f_1 U_1'^{-1} dQ_1.$$

Then it follows that

$$\{L_2, M_1\} = \frac{f_1(f_3 - f_2)}{\Phi}, \qquad \{L_3, M_1\} = \frac{f_1(f_2 g_3 - f_3 g_2)}{\Phi},$$

with analogous relations for M_2 and M_3.

In summary, all brackets between the six functions L_i, L_i' are zero except that

$$\{L_3, L_3'\} = \{L_2, L_2'\} = \{L_1, L_1'\} = 1. \tag{54}$$

Thus the mapping $(x_1, x_2, x_3, p_{x_1}, p_{x_2}, p_{x_3}) \to (L_1, L_2, L_3, L_1', L_2', L_3')$ is canonical.

Note: Standard Hamilton-Jacobi theory gives exactly these same constants of the motion, from a different viewpoint, [1]. Our expression for L_3', for example, is

$$L_3' = \sum_j \int U_j'^{-1} dp_{x_j} = \sum_j M_j,$$

where $U_j = -v_j(x_j) + f_j L_1 + g_j L_2 + L_3$ and $U_j = p_{x_j}^2$. Standard Hamilton-Jacobi theory gives

$$L_3' = \frac{1}{2} \sum_j \int \frac{dx_j}{\sqrt{-v_j + f_j L_1 + g_j L_2 + L_3}} = \sum_j \tilde{M}_j.$$

In the standard theory $\tilde{M}_j = \tilde{M}_j(L_1, L_2, L_3, x_j)$, whereas in our approach $M_j = M_j(L_1, L_2, L_3, p_{x_j})$. In both cases the condition (44) is satisfied. Our approach makes it straightforward to determine exactly when the L'_i are polynomials in the momenta p_{x_j}. It also points out the bracket relations between the M_i and the operators L_j defining the separation, e.g., (43, 48, 49,51, 53).

The generalization to n dimensions is straightforward.

References

[1] V.I. Arnold. Mathematical Methods of Classical Mechanics. (translated by K. Vogtmann and A. Weinstein) *Graduate Texts in Mathematics, 60, Springer-Verlag*, New York, 1978.

[2] Max Karlovini and Kjell Rosquist. Third rank Killing tensors in general relativity. The (1+1)-dimensional case; *Gen.Rel.Grav.* **31**, 1271-1294, (1999).

[3] N.W.Evans. Superintegrability in Classical Mechanics; *Phys.Rev.* **A 41** (1990) 5666;

[4] N.W.Evans. Group Theory of the Smorodinsky-Winternitz System; *J.Math.Phys.* **32**, 3369 (1991).

[5] N.W.Evans. Super-Integrability of the Winternitz System; *Phys.Lett.* **A 147**, 483 (1990).

[6] S.Wojciechowski. Superintegrability of the Calogero-Moser System. *Phys. Lett.* **A 95**, 279 (1983);

[7] L.P.Eisenhart. Enumeration of Potentials for Which One-Particle Schrödinger Equations Are Separable; *Phys.Rev.* **74**, 87 (1948).

[8] J.Friš, V.Mandrosov, Ya.A.Smorodinsky, M.Uhlir and P.Winternitz. On Higher Symmetries in Quantum Mechanics; *Phys.Lett.* **16**, 354 (1965).

[9] J.Friš, Ya.A.Smorodinskii, M.Uhlír and P.Winternitz. Symmetry Groups in Classical and Quantum Mechanics; *Sov.J.Nucl.Phys.* **4**, 444 (1967).

[10] A.A.Makarov, Ya.A.Smorodinsky, Kh.Valiev and P.Winternitz. A Systematic Search for Nonrelativistic Systems with Dynamical Symmetries; *Nuovo Cimento* **A 52**, 1061 (1967).

[11] D.Bonatos, C.Daskaloyannis and K.Kokkotas. Deformed Oscillator Algebras for Two-Dimensional Quantum Superintegrable Systems; *Phys. Rev.* **A 50**, 3700 (1994).

[12] F.Calogero. Solution of a Three-Body Problem in One Dimension; *J.Math.Phys.* **10**, 2191 (1969).

[13] A.Cisneros and H.V.McIntosh. Symmetry of the Two-Dimensional Hydrogen Atom; *J.Math.Phys.* **10**, 277 (1969).

[14] L.G.Mardoyan, G.S.Pogosyan, A.N.Sissakian and V.M.Ter-Antonyan. Elliptic Basis for a Circular Oscillator. *Nuovo Cimento*, **B 88**, 43 (1985), Two-Dimensional Hydrogen Atom: I. Elliptic Bases; *Theor.Math.Phys.* **61**, 1021 (1984); Hidden symmetry, Separation of Variables and Interbasis Expansions in the Two-Dimensional Hydrogen Atom. *J.Phys.*, **A 18**, 455 (1985).

[15] B.Zaslow and M.E.Zandler. Two-Dimensional Analog of the Hydrogen Atom. *Amer. J.Phys.* **35**, 1118 (1967).

[16] J.Hietarinta. Direct methods for the search of the second invariant. *Physics Report*, **147**, 87-154, (1987).

[17] P. Stäckel. Habilitationsschrift, Halle, 1891

[18] L.P. Eisenhart. Separable Systems of Stäckel; *Annals of Math.* (2) **35**, 284-305 (1934).

[19] E.G.Kalnins, W.Miller Jr. and G.S.Pogosyan. Superintegrability and associated polynomial solutions. Euclidean space and the sphere in two dimensions; *J.Math.Phys.* **37**, 6439, (1996).

[20] J.A. Sethian. Level Set Methods and Fast Marching Methods Evolving Interfaces in Computational Geometry, Fluid Mechanics, Computer Vision, and Materials Science; (Chapter 1) Cambridge Monograph on Applied and Computational Mathematics, *Cambridge University Press*, 1999.

[21] R. Courant and D. Hilbert. Methods of Mathematical Physics, Volume II. *Interscience Publishers*, New York, 1962.

[22] D. Zwillinger. Handbook of Differential Equations. *Academic Press*, San Diego, CA, 1989.

[23] E.G. Kalnins and W. Miller. Separable coordinates for three-dimensional complex Riemannian spaces; *J. Diff. Geometry* **14**, 221-236 (1979).

[24] E.G. Kalnins and W. Miller. Killing tensors and variable separation for Hamilton-Jacobi and Helmholtz equations; *SIAM J. Math. Anal.* **11**, 1011-1026 (1980).

[25] W.Miller, Jr. Symmetry and Separation of Variables. *Addison-Wesley Publishing Company*, Providence, Rhode Island, 1977.

[26] E.G. Kalnins, Separation of Variables for Riemannian Spaces of Constant Curvature, *Pitman*, Monographs and Surveys in Pure and Applied Mathematics 28, Longman, Essex, England, 1986.

[27] E.G.Kalnins, J. Kress, W.Miller Jr. and G.S.Pogosyan. Completeness of superintegrability in two-dimensional constant curvature spaces; *J. Phys. A: Math Gen.* **34**, 4705–4720, (2001).