Group classification of systems of non-linear reaction-diffusion equations with general diffusion matrix. II. Diagonal diffusion matrix

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Abstract

Group classification of systems of two coupled nonlinear reaction-diffusion equation with a diagonal diffusion matrix is carried out. Symmetries of diffusion systems with singular diffusion matrix and additional first order derivative terms are described.

1 Introduction

Coupled systems of nonlinear reaction-diffusion equations form the basis of many models of mathematical biology. These systems are widely used in mathematical physics, chemistry and also in social sciences and many other fields. Such reach spectrum of applications stimulates numerous thorough investigations of fundamentals of these equations theory.

In the present paper we continue group classification of systems of reaction-diffusion equations with general diffusion matrix

$$\begin{aligned} &u_t - \Delta(A^{11}u + A^{12}v) = f^1(u,v),\\ &v_t - \Delta(A^{21}u + A^{22}v) = f^2(u,v) \end{aligned} \tag{1}$$

where u and v are function of $t, x_1, x_2, \dots, x_m$, A^{11}, A^{12}, A^{21} and A^{22} are real constants and Δ is the Laplace operator in R^m.

Up to linear transformations of functions u, v and f_1, f_2 it is sufficient to restrict ourselves to such diffusion matrices (i.e., matrices whose elements are $A^{11}, \cdots, A^{22}$) which are diagonal, triangular, or are sums of the unit and antisymmetric matrices. In the last case (1) can be reduced to a single equation for a complex function which we call generalized complex Ginzburg-Landau (CGL) equation. Group classification of CGL equations is carried out in paper [1].

In the present paper we classify equations (1) with diagonal diffusion matrix. Without loss of generality such equations can be written as

$$\begin{aligned} &u_t - \Delta u = f^1(u,v),\\ &v_t - a\Delta v = f^2(u,v) \end{aligned} \tag{2}$$

where a is a constant.

Apparently the first attempt of group classification of equations (2) was made by Danilov [2]. We will show that the results present in [2] are far from completeness.

Group classification of equations (2) with general diffusion matrix was announced in [3] and presented in [4]. Unfortunately due to typographical errors the tables with classification results present there are poorly readable (see [1] for additional comments).

Symmetries of systems of reaction-diffusion equations with diagonal diffusion matrix (i.e., of systems (2)) where studied in papers [5], [6]. We will see in the following that classification results given in [5], [6] are incomplete and include many equivalent cases treated as non-equivalent.

It is practically impossible to enumerate all fields of applications of systems (1). We restrict ourselves to few examples only.

- The Jackiw-Teitelboim model of two-dimension gravity with the non-relativistic gauge [7]

$$\begin{aligned} &u_t - u_{xx} = 2ku - 2u^2v = 0, \\ &v_t + v_{xx} = 2uv^2 - 2kv = 0. \end{aligned} \tag{3}$$

  Symmetries of equations (3) were investigated in paper [8]. In the following we correct and complete results obtained in [8].

- The primitive predator-prey system which can be defined by [9]

$$u_t - Du_{xx} = -uv, \qquad v_t - \lambda D v_{xx} = uv. \tag{4}$$

  also appears as an particular subject of our analysis.

- The $\lambda - \omega$ reaction-diffusion system [10]

$$u_t = D\Delta u + \lambda(R)u - \omega(R)v, \qquad v_t = D\Delta v + \omega(R)u + \lambda(R)v, \tag{5}$$

  where $R^2 = u^2 + v^2$, is widely used in studies of reaction-diffusion models, in particular, to describe spiral waves phenomena [11] .

  Symmetries of equations (5) were studied in paper [12]. We shall add the results [12] in the following.

Thus the problem of group classification of equations (2) is still actual and we will present its solution here. In addition, for the first time we shall classify equations (2) with non-invertible diffusion matrix (i.e., equations (2) when parameter a is equal to zero) and also the following equations which include the first derivative terms:

$$\begin{aligned} &u_t - \Delta u = f^1(u, v), \\ &v_t - p_\mu v_{x_\mu} = f^2(u, v) \end{aligned} \tag{6}$$

where $u_{x_\mu} = \frac{\partial u}{\partial x_\mu}$, p_μ are arbitrary constants and summation from 1 to m is imposed over the repeated index μ. Moreover, without loss of generality one can set

$$p_1 = p_2 = \cdots = p_{m-1} = 0, \ p_m = p. \tag{7}$$

In the case $p \equiv 0$ equation (6) reduces to (2) with $a = 0$. The latest equation is used in such popular models of mathematical biology as the FitzHung-Naguno [13] and Rinzel-Keller [14] ones.

2 Equivalence transformations

The problem of group classification of equations (2), (6) will be solved up to equivalence transformations. Clear definition of these transformations is one of the main points of any classification procedure.

We say the equations

$$\begin{aligned} &\tilde{u}_t - \Delta\tilde{u} = \tilde{f}^1(u, v), \\ &\tilde{v}_t - a\Delta\tilde{v} = \tilde{f}^2(u, v) \end{aligned} \tag{8}$$

be equivalent to (2) if there exist an invertible transformation $u \to \tilde{u} = G(t, x, u, v)$, $v \to \tilde{v} = \Phi(t, x, u, v)$, $t \to \tilde{t} = T(t, x, u, v)$, $x \to \tilde{x} = X(t, x, u, v)$ and $\tilde{f}_\alpha \to \tilde{f} = F_\alpha(u, t, x, f)$ which connects (2) with (8). In other words the equivalence transformations should keep the general form of equation (2) but can change concrete realization of non-linear terms f^1 and f^2.

The group of equivalence transformations for equation (2) can be found using the classical Lie approach and treating f^1 and f^2 as additional dependent variables. In addition to the obvious symmetry transformations

$$t \to t' = t + a, \quad x_\mu \to x'_\mu = R_{\mu\nu}x_\nu + b_\mu \tag{9}$$

where a, b_μ and $R_{\mu\nu}$ are arbitrary parameters satisfying $R_{\mu\nu}R_{\mu\lambda} = \delta_{\mu\lambda}$, this group includes the following transformations

$$\begin{aligned} &u_b \to K^{bc}u_c + b_b, \quad f^b \to \lambda^2 K^{bc} f^c, \\ &t \to \lambda^{-2}t, \quad x_b \to \lambda^{-1}x_b \end{aligned} \tag{10}$$

and

$$\begin{aligned} &u_b \to \tilde{K}^{bc}u_c, \quad f^b \to a\tilde{K}^{bc} f^c, \\ &t \to a^{-1}t, \quad x_b \to a^{-\frac{1}{2}}x_b, \quad a \neq 0 \end{aligned} \tag{11}$$

where indices b, c take values 1 and 2, K^{bc} and $\tilde{K}^{bc}$ are elements of an invertible constant matrices K and $\tilde{K}$ respectively, moreover, K commutes and $\tilde{K}$ anticommutes with A; $\lambda \neq 0$ and b_a are arbitrary constants, and we use the temporary notations $u = u_1, v = u_2$. In this Section and Sections 3, 4, 7 we use for dependent variables both notations u, v and u_1, u_2 simultaneously.

If parameter a is equal to 1 then K is an arbitrary 2×2 invertible matrix, and equivalence transformations (11) do not exist. For $a \neq 1$ matrices K and $\tilde{K}$ have the following form

$$K = \begin{pmatrix} K_{11} & 0 \\ 0 & K_{22} \end{pmatrix}, \quad \tilde{K} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Transformation (11) reduce to the change $a \to 1/a$ in the related matrix A, i.e., to scaling of arbitrary constant a. Thus without loss of generality we can restrict ourselves to the following values of a:

$$1.\ a = 0, \quad 2.\ 0 < a < 1, \quad 3.\ a = 1. \tag{12}$$

It is possible to show that there is no more extended equivalence relations valid for arbitrary nonlinearities f^1 and f^2. However, if functions f^1, f^2 are fixed, the invariance group in general is more extended. In addition to transformations (10) and (11) it includes symmetry transformations which does not change the form of equation (2). Moreover, for some classes of functions f^1, f^2 equation (2) admits additional equivalence transformations (AET) which do not belong to symmetry transformations.

In spite of the fact that we search for AET *after* description of symmetries of equations (2) and specification of functions f^1, f^2, for convenience we present the list of the additional equivalence transformations in the following formulae:

$$\begin{array}{ll}
1. & u \to \exp(\omega t)u, \quad v \to \exp(\rho t)v, \\
2. & u \to u + \omega t, \ v \to v, \\
3. & u \to u, \ v \to v + \rho t, \\
4. & u \to u + \rho t, \ v \to v \exp(\rho t), \\
5. & u \to \exp(\omega t)u, \ v \to v + \omega t, \\
6. & u \to u, \ v \to v + \rho t u, \\
7. & u \to \exp(\omega t)u, \ v \to v + \omega \frac{t^2}{2}, \\
8. & u \to \exp(\omega t)u, \ v \to v + \kappa t u + \rho \frac{t^2}{2}, \\
9. & u \to u, \ v \to v - \rho t u + \rho\lambda \frac{t^2}{2}, \\
10. & u \to \exp(\rho t)u, \ v \to v - \kappa\rho t, \\
11. & u \to \exp(\rho t)u, \ v \to \exp(\rho t)\left(v + \varepsilon \frac{t^2}{2} u\right), \\
12. & u \to u + \rho t, \ v \to v - \rho t, \\
13. & u \to u + \rho t, \ v \to e^{-\frac{\rho}{\nu} t} v, \\
14. & u \to u + \rho t, \ v \to v + \rho t u + \rho \frac{t^2}{2}, \\
15. & u \to u \cos\omega t - v \sin\omega t, \ v \to v \cos\omega t + u \sin\omega t.
\end{array} \tag{13}$$

Here the Greek letters denote parameters whose values are specified in the tables presented below. We stress once more that equivalence transformations (13) are valid only for some concrete non-linearities which will be specified in the following.

3 Symmetries and classifying equations

As usual, we will search for symmetries of equations (2) and (6) with respect to continuous groups of transformations using the infinitesimal approach. Using the Lie algorithm or its simplified version proposed in [4] one can find the determining equations for coefficient functions $\eta,\ \xi_a,\ \pi^b$ of generator X of the symmetry group:

$$X = \eta\partial_t + \xi^a\partial_{x_a} - \pi^b\partial_{u_b} \tag{14}$$

and classifying equations for non-linearities f^1 and f^2. We will not reproduce the related routine calculations but present the general form of symmetry X for equation (2) with $a \neq 0$ found in [4] (see also [1]) :

$$\begin{array}{l} X = \lambda K + \sigma_\mu G_\mu + \omega_\mu \hat{G}_\mu + \mu D - (C^{ab}u_b + B^a)\partial_{u_a} \\ +\Psi^{\mu\nu}x_\mu\partial_{x_\nu} + \nu\partial_t + \rho_\mu\partial_{x_\mu} \end{array} \tag{15}$$

where the Greek letters denote arbitrary constants, B^a and are functions of t, x and t respectively, matrix whose elements are C^{ab} have to commute with A, and

$$C^{ab}A^{bk} - A^{ab}C^{bk} = 0, \tag{16}$$

and

$$\begin{array}{l} K = t(t\partial_t + x_\mu\partial_{x_\mu}) - \frac{x^2}{4}(A^{-1})^{ab}u_b\partial_{u_a} - \frac{tm}{2}u_a\partial_{u_a}, \\ G_\mu = t\partial_{x_\mu} + \frac{1}{2}x_\mu(A^{-1})^{ab}u_b\partial_{u_a}, \\ \hat{G}_\mu = e^{\gamma t}\left(\partial_{x_\mu} + \frac{1}{2}\gamma x_\mu(A^{-1})^{ab}u_b\partial_{u_a}\right), \\ D = t\partial_t + \frac{1}{2}x_\mu\partial_{x_\mu}. \end{array} \tag{17}$$

Here A^{ab} and $(A^{-1})^{ab}$ are elements of matrix A and matrix inverse to A respectively.

If $a = 0$ then the related generator X again has the form (15) where however $\lambda = \sigma_\mu = \omega_\mu = 0$.

Equation (2) admits symmetry (15) iff the following classifying equations for f^1 and f^2 are satisfied [1]:

$$\begin{array}{l} (2\lambda t + \mu)f^a + \left(\frac{1}{4}\lambda x^2 + \sigma_\mu x_\mu + \gamma e^{\gamma t}\omega_\mu x_\mu\right)(A^{-1})^{ab}f^b + \frac{1}{2}tmf^a \\ +C^{ab}f^b + C^{ab}_t u_b + \frac{\lambda m}{2}\left(u^a - (A^{-1})^{ab}u_b\right) + B^a_t - \Delta A^{ab}B^b \\ = (B^s + C^{sb}u_b + \left(\frac{1}{4}\lambda x^2 + \sigma_\mu x_\mu + \gamma e^{\gamma t}\omega_\mu x_\mu\right)(A^{-1})^{sk}u_k)f^a_{u_s}. \end{array} \tag{18}$$

In other words, to make group classification of systems (2) means to find all non-equivalent solutions of equations (18) and to specify the related symmetries (15) [4].

We stress that relations (15)-(18) are valid for group classification of systems (2) including *arbitrary* number n of dependent variables $u = (u_1, u_2, \dots u_n)$ provided the related $n \times n$ matrix A be invertible [4]. In this case indices a, b, s, k in (15)-(18) run over the values $1, 2 \dots n$.

Consider now equation (6) and the related symmetry operator (14). The determining equations for η, ξ^μ and π^a are easily obtained using the standard Lie algorithm and have the following form

$$\begin{array}{l}\eta_{tt}=\eta_{x_\mu}=\eta_u=\eta_v=0, \quad \xi^\mu_t=\xi^\mu_u=\xi^\mu_v=0,\\ \pi^a_{uv}=0, \quad \pi^a_{x_\mu u}+\pi^a_{x_\mu v}=0, \quad \pi^1_v=\pi^2_u=0; \quad \pi^1_u-\pi^2_v=\frac{1}{2}\eta_t, \ \text{ if } \ p\neq 0;\\ \xi^\mu_{x_\nu}+\xi^\nu_{x_\mu}=-\delta^{\mu\nu}\eta_t, \ \mu\neq m\end{array} \qquad (19)$$

where subscripts denote derivatives w.r.t. the corresponding independent variable, i.e., $\eta_t=\frac{\partial\eta}{\partial t}$, $\xi^\mu_{x_\nu}=\frac{\partial\xi^\mu}{\partial x_\nu}$, etc.

Integrating system (19) we obtain the general form of operator X:

$$X=\nu\partial_t+\rho_\nu\partial_\nu+\Psi^{\sigma\nu}\partial_\nu x_\sigma+\mu D-B^1\partial_u-B^2\partial_v-Fu\partial_u-Gv\partial_v; \qquad (20)$$

$$F-G=\frac{1}{2}\mu \ \text{ if } p\neq 0 \qquad (21)$$

where B^1, B^2 are functions of (t,x), F and G are functions of t and summation over the indices σ,ν is assumed with $\sigma,\nu=1,2,\cdots,n-1$.

The classifying equations for non-linearities f^1 and f^2 reduce to the following ones

$$\begin{array}{l}(\mu+F)f^1+F_tu_1+(\partial_t-\Delta)B^1\\ =\left(B^1\partial_u+B^2\partial_v+Fu\partial_u+Gv\partial_v\right)f^1,\end{array} \qquad (22)$$

$$\begin{array}{l}(\mu+G)f^2+G_tu_2+B^2_t-pB^1_{x_m}\\ =\left(B^1\partial_u+B^2\partial_v+Fu_1\partial_u+Gu_2\partial_v\right)f^2.\end{array} \qquad (23)$$

Relations (20)–(23) are valid for both cases $p\neq 0$ and $p=0$ (in the last case condition (21) should be omitted). Solving (22), (23) we shall specify both the coefficients of infinitesimal operator (20) and the related non-linearities f^1 and f^2.

It is obvious that the widest spectrum of symmetries appears in the case when the parameter a is equal to 1, in as much as the corresponding relation (16) does not impose any restrictions for functions C^{ab}. Quite the contrary, equations (2) with $a\neq 1$ and especially (6) admit relatively small variety of symmetries.

4 Classification of symmetries

Following [1] we specify basic, main and extended symmetries for the analyzed systems of reaction-diffusion equations.

Basic symmetries are nothing but generators of transformations (9) forming the kernel of a symmetry group, i.e.,

$$P_0 = \partial_t, \quad P_\lambda = \partial_\lambda, \quad J_{\mu\nu} = x_\mu \partial_{x_\nu} - x_\nu \partial_{x_\mu}. \tag{24}$$

Main symmetries form an important subclass of general symmetries (15) and have the following form

$$\tilde{X} = \mu D + C^{ab} u_b \partial_{u_a} + B^a \partial_{u_a}. \tag{25}$$

In accordance with the analysis present in [1] complete description of general symmetries (15) can be obtained using the following steps:

- Find all main symmetries (25), i.e., solve equations (18) for $\Psi^{\mu\nu} = \nu = \rho_\nu = \sigma_\nu = \omega_\nu = 0$:

$$(\mu\delta^{ab} + C^{ab}) f^b + C_t^{ab} u_b + B_t^a - \Delta A^{ab} B^b = (C^{nb} u_b + B^n) f^a_{u_n}. \tag{26}$$

- Specify all cases when the main symmetries can be extended, i.e., at least one of the following relations are satisfied:

$$(A^{-1})^{ab} f^b = (A^{-1})^{nb} u_b f^a_{u_n}, \tag{27}$$

$$(A^{-1})^{kb} (f^b + \gamma u^b) = (A^{-1})^{ab} u_b f^k_{u_a}. \tag{28}$$

  or if equation (27) is satisfied together with the following condition:

$$\tilde{C}^{kb} f^b + \tilde{C}_t^{kb} u_b + \frac{m}{2}\left(u^k - (A^{-1})^{kb} u_b\right) + \tilde{B}_t^k - (\Delta A^{kb} \tilde{B}^b) = (\tilde{C}^{ab} u_b + \tilde{B}^a) f^k_{u_a} \tag{29}$$

  where

$$\tilde{C}^{ab} = C^{ab} - \nu \int C^{ab} dt, \quad \tilde{B}^a = B^a - \nu \int B^a dt.$$

  If relations (27), (28) or (29) are valid then the corresponding system (2) admits symmetry $G_\alpha, \widehat{G}_\alpha$ or K correspondingly (see (17) for definitions).

- This algorithm is valid for classification of systems (2) including arbitrary number n of dependent variables $u = (u_1, u_2, \ldots u_n)$ provided the related $n \times n$ matrix A be invertible.

- When classifying equations (6) the second step in not needed in as much as in accordance with (20) these equations admit only basic and mane symmetries.

In the following sections we find main and extended symmetries for classified equations. For clarity we start with group classification of systems (6) which is more simple technically and present rather detailed calculations. Then we consider equations (2) and present classification results without technical details.

5 Algebras of main symmetries for equation (6)

To describe main symmetries we use the trick discussed in [1], i.e., make *a priori* classification of low dimension algebras of these symmetries. In accordance with (20) any symmetry generator extending algebra (24) has the following form

$$X = \mu D - B^1\partial_u - B^2\partial_v - Fu\partial_u + \left(\frac{\mu}{2} - F\right) v\partial_v. \tag{30}$$

Let X_1 and X_2 be operators of the form (30) then the commutator $[X_1, X_2]$ is also a symmetry whose general form is given by (30). Thus operators (30) form a Lie algebra which we denote as $\mathcal{A}$.

Let us specify algebras $\mathcal{A}$ which can appear in our classification procedure. First consider one-dimensional $\mathcal{A}$, i.e., suppose that equation (6) admits the only symmetry of the form (30). Then any commutator of operator (24) with (30) should be equal to a linear combination of operators (24) and (30). Using this condition we come to the following possibilities only:

$$\begin{aligned} &X = X_1 = \mu D - \alpha_1\partial_u - \alpha_2\partial_v - \beta u\partial_u - (\beta - \tfrac{\mu}{2})v\partial_v,\\ &X = X_2 = e^{\nu t}(\alpha_1\partial_u + \alpha_2\partial_v + \beta u\partial_u + \beta v\partial_v),\\ &X = X_3 = e^{\nu t + \rho\cdot x}(\alpha^1\partial_u + \alpha^2\partial_v) \end{aligned} \tag{31}$$

where the Greek letters again denote arbitrary parameters and $\rho \cdot x = \rho_\mu x_\mu$.

The next step is to specify all non-equivalent sets of arbitrary constants in (31) using the equivalence transformations (10).

If the coefficient for $u\partial_u$ (or $v\partial_v$) is non-zero then translating u (v) we reduce to zero the related coefficient α_1 (α_2) in X_1 and X_2; then scaling u (v) we can reduce to ± 1 all non-zero α_a in (31). In addition, all operators (31) are defined up to constant multipliers. Using these simple arguments we come to the following non-equivalent versions of operators (31) belonging to one-dimensional algebras $\mathcal{A}$:

$$\begin{aligned} &X_1^{(1)} = 2\mu D - u\partial_u + (\mu - 1)v\partial_v,\\ &X_1^{(2)} = 2D + v\partial_v + \nu\partial_u,\ X_1^{(3)} = 2D - u\partial_u - \partial_v,\\ &X_2^{(\nu)} = e^{\nu t + \rho_2\cdot x}(u\partial_u + v\partial_v);\quad X_3^{(1)} = e^{\sigma_1 t + \rho_1\cdot x}(\partial_u + \partial_v),\\ &X_3^{(2)} = e^{\sigma_2 t + \rho_2\cdot x}\partial_u,\ X_3^{(3)} = e^{\sigma_3 t + \rho_3\cdot x}\partial_v. \end{aligned} \tag{32}$$

To describe *two-dimensional* algebras $\mathcal{A}$ we represent one of the related basis element X in the general form (30) and calculate the commutators

$$Y = [P_0, X] - 2\mu P_0, \quad Z = [P_0, Y], \quad W = [X, Y]$$

where P_0 is operator given in (24). After simple calculations we obtain

$$\begin{aligned} &Y = F_t(u\partial_u + v\partial_v) + B^1_t\partial_u + B^2_t\partial_v, \quad Z = F_{tt}(u\partial_u + v\partial_v) + B^1_{tt}\partial_u + B^2_{tt}\partial_v,\\ &W = 2\mu t Z + \mu x_b(B^1_{tx_b}\partial_u + B^2_{tx_b}\partial_v. \end{aligned} \tag{33}$$

By definition, Y, Z and W belong to $\mathcal{A}$. Let $F_t \neq 0$ than it follows from (33) that

$$\mu \neq 0: \;\; B^a_{tt} = F_{tt} = B^a_{tb} = 0, \tag{34}$$

$$\mu = 0: \;\; F_{tt} = \alpha F_t + \gamma^a B^a_t, \quad B^a_{tt} = \gamma^a F_t + \beta^{ab} B^b_t. \tag{35}$$

Starting with (34) we conclude that up to translations of t the coefficients F and B_a have the following form

$$F = \sigma t \text{ or } F = \beta; \;\; B^a = \nu^a t + \alpha^a \text{ if } \mu \neq 0.$$

If $F = \sigma t$ then the change

$$u_a \rightarrow u_a e^{-\sigma t} - \frac{\nu_a}{\mu} t \tag{36}$$

reduces the related operator (20) to X_1 of (31) for $\beta = 0$.

The choice $F = \beta$ corresponds to the following operator (30)

$$X = X_4 = X_1 - 2t(\alpha^1 \partial_u + \alpha^2 \partial_v) \tag{37}$$

where X_1 is given in (31).

Thus if one of basis elements of two dimension algebra $\mathcal{A}$ is of general form (30) with $\mu \neq 0$ then it can be reduced to X_1 with $\beta = 0$ or to generator (37). We denote such basis element as e_1. Without loss of generality the second basis element e_2 of $\mathcal{A}$ is a linear combination of operators $X_2^{(\nu)}$ and $X_3^{(a)}$ (32). Going over possible pairs (e_1, e_2) and requiring $[e_1, e_2] = \alpha_1 e_1 + \alpha_2 e_2$ we come to the following two dimensional algebras

$$\begin{array}{l} A_1 =< 2D + v\partial_v, X_2^{(0)} >, \quad A_2 =< X_1^{(2)}, X_3^{(3)} >, \\ A_3 =< X_1^{(3)}, X_3^{(3)} >, \quad A_4 =< X_1^{(1)}, \quad X_3^{(3)} >, \\ A_5 =< X_1^{(1)}, X_3^{(3)} >, \quad A_6 =< 2D + 2v\partial_v + u\partial_u + \nu t \partial_v, X_3^{(2)} > \\ A_7 =< 2D + 2u\partial_u + 3v\partial_v + 3\nu t\partial_u, \quad X_3^{(1)} > . \end{array} \tag{38}$$

The form of basis elements in (38) is defined up to transformations (36) (10).

If $\mathcal{A}$ does not include operators (30) with non-trivial parameters μ then in accordance with (36) its elements are of the following form

$$e_a = F_{(a)} \left(u\partial_u + v\partial_v \right) + B^1_{(a)} \partial_u + B^2_{(a)} \partial_v, \;\; a = 1, 2 \tag{39}$$

where $F_{(\alpha)}$ and $B^1_{(a)}$, $B^2_{(a)}$ are solutions of (35).

Formulae (38), (39) define all non-equivalent two-dimensional algebras $\mathcal{A}$ which have to be considered as possible symmetries of equations (6). We will see that asking for invariance of (6) w.r.t. these algebras the related arbitrary functions f^a are defined up to arbitrary constants, and it is impossible to make further specification of these functions by extending algebra $\mathcal{A}$.

6 Group classification of equations (6)

To classify equations (6) which admit one- and two- dimension extensions of the basis invariance algebra (24) it is sufficient to solve determining equations (22) for f^a with *known* coefficient functions B^a and F of symmetries (30). These functions are easily found comparing (20) with (32), (38) and (39).

Let us present an example of such calculation which corresponds to algebra A_1 whose basis elements are $X_1 = 2t\partial_t + x_a\partial_{x_a} + v\partial_v$ and $X_2^{(0)} = u\partial_u + v\partial_v$, refer to (38). Operator $X_2^{(0)}$ generates the following form of equation (22):

$$f^a = (u\partial_u + v\partial_v)\, f^a, \quad a = 1, 2$$

whose general solution is

$$f^1 = uF_1\left(\frac{v}{u}\right), \quad f^2 = uF_2\left(\frac{v}{u}\right). \tag{40}$$

Here F_1 and F_2 are arbitrary functions of $\frac{v}{u}$.

Equations (6) with non-linearities (40) admit symmetry $X_2^{(0)}$. In order this equation be invariant w.r.t. X_1 also, functions f^1, f^2 have to satisfy equation (22) with $F = 0$, i.e.,

$$f^1 = -uf^1_u; \quad f^2 = -\frac{1}{2}uf^2_u. \tag{41}$$

It follows from (40), (41) that

$$f^1 = \alpha u^3 v^{-2}, \quad f^2 = \lambda u^2 v^{-1}. \tag{42}$$

Thus equation (6) admits symmetries $X_0^{(2)}$ and X_1 which form algebra A_1 (38) provided f^1 and f^2 are functions given in (42). These symmetries are defined up to arbitrary constants α and λ, if one of them is nonzero, than it can be reduced to $+1$ or -1 by scaling independent variables. In addition, using MULIE software we verify that this equation does not admit more extended symmetries.

In analogous way we solve equations (22) corresponding to other symmetries indicated in (32) and (38). For one-dimension algebras (32) the related non-linearities f^1 and f^2 are defined up to arbitrary functions F_1 and F_2 while for two dimension algebras (38) functions f^1 and f^2 are defined up to two integration constants. Algebras (39) either lead to incompatible equations (22) or correspond to linear functions f^1 and f^2 which are not considered here. We shall not reproduce the related rather routine calculations but present their results in Table 1 where Greek letters denote arbitrary parameters. Moreover, without loss of generality we can restrict ourselves to $\lambda = 0, 1$, $\alpha, \beta = 0, \pm 1$ (in items 11 and 12 $\alpha = 0, 1$ if ν is half-integer).

Table 1. Non-linearities and symmetries for equation (6)

No	Non-linearities	Arguments of F_1, F_2	Symmetries
1.	$f^1 = u_1^{2\nu+1} F_1,$ $f^2 = u^{\nu+1} F_2$	$vu^{\nu-1}$	$2\nu D - u\partial_u$ $+(\nu-1)v\partial_v$
2.	$f^1 = F_1 u_2^{-2},$ $f^2 = F_2 u_2^{-1}$	$u - \mu \ln v$	$2D + v\partial_v + \mu\partial_u$
3.	$f^1 = u(F_1 + \lambda \ln u),$ $f^2 = v(F_2 + \lambda \ln u)$	$\frac{v}{u}$	$e^{\lambda t}(u\partial_u + v\partial_v)$
4.	$f^1 = u_1^3 F_1,$ $f^2 = u_1^2 F_2$	$v - \ln u$	$2D - u\partial_u - \partial_v$
5.	$f^1 = \alpha u + F_1,$ $f^2 = F_2 - \nu p u$	v	$e^{\lambda t + \nu x_m} \tilde{\Psi}_\mu(\tilde{x})\partial_u,$ $\mu = \lambda - \nu^2 - \alpha$
6.	$f^1 = F_1,$ $f^2 = F_2 + \nu v$	u	$e^{\nu t}\Psi(x)\partial_v$
7.	$f^1 = \alpha u + F_1$ $f^2 = F_2 + (\lambda + p\nu)v$	$u - v$	$e^{(\nu p+\rho)t + \nu x_m} \tilde{\Psi}_\mu(\tilde{x})(\partial_u + \partial_v),$ $\mu = \nu p + \rho - \nu^2 - \alpha$
8.	$f^1 = \alpha u_1^3 u_2^{-2},$ $f^2 = \beta u_1^2 u_2^{-1}$		$2D + v\partial_v, \quad u\partial_u + v\partial_v$
9.	$f^1 = \alpha e^{-2u},$ $f^2 = \lambda e^{-u}$		$2D + v\partial_v + \partial_u, \quad \Psi(x)\partial_v$
10.	$f^1 = \lambda e^{3v},$ $f^2 = \alpha e^{2v}$		$2D - u\partial_x u - \partial_v,$ $e^{(\lambda+\mu)t}\tilde{\Psi}_\mu(\tilde{x})\partial_u$
11.	$f^1 = \alpha u^{2\nu+1},$ $f^2 = \lambda u^{\nu+1}$		$\nu 2D - u\partial_u + (\nu-1)v\partial_v,$ $\Psi(x)\partial_v$
12.	$f^1 = \lambda v^{3\nu-2},$ $f^2 = \alpha u_2^{2\nu-1}$		$2(\nu-1)D - \nu u\partial_u - v\partial_v,$ $\tilde{\Psi}_\nu(\tilde{x})\partial_u$
13.	$f^1 = \frac{\alpha}{u}, \quad f^2 = \ln u$		$2D + 2v\partial_v + u\partial_u + t\partial_v,$ $\Psi(x)\partial_v$
14.	$f^1 = \ln v, \quad f^2 = \alpha v^{\frac{1}{3}}$		$2D + 2u\partial_u + 3v\partial_v + 3t\partial_u,$ $\tilde{\Phi}_\mu(t,\tilde{x})\partial_u$

Here D is the dilatation operator given in (17), $\tilde{x} = (x_1, x_2, \cdots, x_{m-1})$, $\Psi(x)$ is an arbitrary function of spatial variables; $\tilde{\Psi}_\mu(\tilde{x})$ and $\tilde{\Phi}_\mu(t,\tilde{x})$ are solutions of the following equations

$$\tilde{\Delta}\tilde{\Psi}_\mu(\tilde{x}) = \mu\tilde{\Psi}_\mu(\tilde{x}), \ (\tfrac{\partial}{\partial t} - \tilde{\Delta})\tilde{\Phi}_\mu(\tilde{x}) = \mu\tilde{\Phi}_\mu(\tilde{x}),$$
$$\tilde{\Delta} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_{m-1}^2}.$$

We notice that non-linearities present in Items 1 and 11 are defined for all values of ν including the case $\nu = 0$ when the dilatation symmetries are reduced to scaling of dependent components only.

7 Algebras of main symmetries for equation (2)

Let us start with group classification of systems of coupled reaction-diffusion equations (2) and consider all types of the corresponding matrices A . In accordance with the plane outlined in Section 4 we first describe the main symmetries generated by operators (25) and then indicate extensions of these symmetries.

Like in Sections 5, 6 the first step of our analysis consists in description of realizations of Lie algebras $\mathcal{A}$ generating basic symmetries. The general form of basis elements of $\mathcal{A}$ is given by relation (25).

Following [1] we first specify all non-equivalent terms

$$N = C^{ab} u_b \partial_{u_a} + B^a \partial_{u_a}. \tag{43}$$

where summation from 1 to 2 is imposed over the repeated indices and we again use the notations $u_1 = u, u_2 = v$.

Let (43) is a basis element of a one-dimensional invariance algebra $\mathcal{A}$ then commutators of N with P_0 and P_a should be equal to a linear combination of N and operators (24). This condition presents three the following possibilities [1]:

$$\begin{array}{l} 1.\ \ C^{ab} = \mu^{ab}, \quad B^a = \mu^a, \\ 2.\ \ C^{ab} = e^{\lambda t}\mu^{ab}, \quad B^a = e^{\lambda t}\mu^a, \\ 3.\ \ C^{ab} = 0, \quad B^a = e^{\lambda t + \omega \cdot x}\mu^a \end{array} \tag{44}$$

where μ^{ab}, μ^a, λ, and ω are constants.

Like in [1] to classify all non-equivalent symmetries (44) we use their isomorphism with 3×3 matrices of the following form

$$g = \begin{pmatrix} 0 & 0 & 0 \\ \mu^1 & \mu^{11} & \mu^{12} \\ \mu^2 & \mu^{21} & \mu^{12} \end{pmatrix}. \tag{45}$$

Equations (2) admit equivalence transformations (10).The corresponding transformation for matrix (45) are

$$g \to g' = UgU^{-1} \tag{46}$$

where U is a 3×3 matrix of the following special form

$$U = \begin{pmatrix} 1 & 0 & 0 \\ b^1 & K^{11} & K^{12} \\ b^2 & K^{21} & K^{22} \end{pmatrix}. \tag{47}$$

were K^{ab} are the same parameters as in (10), (11).

Let us consider equation (2) with a diagonal matrix A (versions 1, 2 of (12)) and find the related low-dimension algebras $\mathcal{A}$. In this case matrix (45) and the equivalence transformation matrix (47) reduce to the forms

$$g = \begin{pmatrix} 0 & 0 & 0 \\ \mu^1 & \mu^{11} & 0 \\ \mu^2 & 0 & \mu^{22} \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 & 0 \\ b^1 & K^1 & 0 \\ b^2 & 0 & K_2 \end{pmatrix}. \tag{48}$$

Up to equivalence transformations (46) there exist three non-equivalent matrices (48), namely

$$g_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 0 & 0 & 0 \\ \lambda & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{49}$$

In accordance with (43)-(**??**) the related symmetry operator can be represented in one of the following forms

$$X_1 = \mu D - 2(g_a)_{bc}\tilde{u}_c\partial_{u_b}, \; X_2 = e^{\lambda t}(g_a)_{bc}\tilde{u}_c\partial_{u_b} \tag{50}$$

or

$$X_3 = e^{\lambda t + \omega\cdot x}\left(\partial_{u_2} + \mu\partial_{u_1}\right). \tag{51}$$

Here $(g_a)_{bc}$ are elements of matrices (49), $b, c =$0, 1, 2, $\tilde{u} =$ column (u_0, u_1, u_2), $u_0 = 1$.

Formulae (50) and (51) give the principal description of one-dimension algebras A for equation (2) with $a \neq 1$.

To describe two-dimension algebras $\mathcal{A}$ we classify matrices g (48) forming two-dimension Lie algebras. Choosing a basis element in one of the forms given in (49) and the other element in the general form (48) we find that up to equivalence transformations (45) there exist six algebras $< e_1, e_2 >$:

$$A_{2,1} = \{\tilde{g}_1, g_4\}, \quad A_{2,2} = \{\tilde{g}_1, \tilde{g}_3\}, \quad A_{2,3} = \{g_5, \tilde{g}_3\}, \tag{52}$$

$$A_{2,4} = \{g_1, g_5\}, \quad A_{2,5} = \{g_1', g_3\}, \quad A_{2,6} = \{g_2, \tilde{g}_3\} \tag{53}$$

where $\tilde{g}_1 = g_1|_{\lambda=0}$, $g_1' = g_1|_{\lambda=1}$, $\tilde{g}_3 = g_3|_{\lambda=0}$, and

$$g_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_5 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{54}$$

Algebras (52) are Abelian while algebras (53) are characterized by the following commutation relations:

$$[e_1, e_2] = e_2 \tag{55}$$

where e_1 is the first element given in the brackets (53), i.e., for $A_{2,4}$ $e_1 = g_1$, etc.

Using (52), (53) and applying arguments analogous to those which follow equations (31) we easily find pairs of operators (25) forming Lie algebras. Denoting

$$\hat{e}_\alpha = (e_\alpha)_{ab}\tilde{u}_b \frac{\partial}{\partial u_a}, \quad \alpha = 1, 2$$

we represent them as follows:

$$< \mu D - \hat{e}_1, \nu D - \hat{e}_2 >, \ < F_1\hat{e}_1 + G_1\hat{e}_2, \ F_2\hat{e}_1 + G_2\hat{e}_2 > \tag{56}$$

for e_1, e_2 belonging to algebras (52);

$$< \mu D - \hat{e}_1, \hat{e}_2 > \tag{57}$$

for e_1, e_2 belonging to algebras (52) and (53);

$$< \mu D + \hat{e}_1 + \nu t\hat{e}_2, \hat{e}_2 > \tag{58}$$

for e_1, e_2 belonging to algebras (53).

Here $\{F_1, G_1\}$ and $\{F_2, G_2\}$ are fundamental solutions of the following system

$$F_t = \lambda F + \nu G, \quad G_t = \sigma F + \gamma G \tag{59}$$

with arbitrary parameters $\lambda, \nu, \sigma, \gamma$.

The list (56)-(58) does not includes algebras of the type $< e^{\nu t + \omega\cdot x}\hat{e}, e^{\mu t + \lambda\cdot x}\hat{e} >$ which are incompatible with determining equations (18). All the other two-dimension algebras $\mathcal{A}$ can be reduced to one the form given in (56) -(58) using equivalence transformations (10), (36).

Up to equivalence there exist three realizations of three-dimension algebras in terms of matrices (49), (54):

$$\begin{array}{ll} A_{3,1}: & e_1 = \tilde{g}_1, \ e_2 = g_4, \ \tilde{g}_3, \\ A_{3,2}: & e_1 = g_5, \ e_2 = g_4, \ e_3 = \tilde{g}_3, \end{array} \tag{60}$$

$$A_{3,3}: \quad e_1 = g_1', \ e_2 = g_5, \ e_3 = \tilde{g}_3. \tag{61}$$

Non-zero commutators for matrices (60) and (61) are $[e_2, e_3] = e_3$ and $[e_1, e_\alpha] = e_\alpha (\alpha = 2, 3)$ respectively. The algebras of operators (25) corresponding to realizations (60) and (61) are of the following general forms:

$$< \mu D - \hat{e}_1,\ \nu D - \hat{e}_2, \hat{e}_3 >,\ < \hat{e}_1,\ D + \hat{e}_2 + \mu t \hat{e}_3,\ \hat{e}_3 > \tag{62}$$

and

$$\begin{array}{l} < \mu D - \hat{e}_1,\ \hat{e}_2,\ \hat{e}_3 >,\ < D + \hat{e}_1 + \nu t \hat{e}_2,\ \hat{e}_2,\ \hat{e}_3 >, \\ < D + \hat{e}_1 + \nu t \hat{e}_3,\ \hat{e}_3,\ \hat{e}_2 >,\ < \hat{e}_1,\quad F_1 \hat{e}_2 + G_1 \hat{e}_3,\ F_2 \hat{e}_2 + G_2 \hat{e}_3 > \end{array} \tag{63}$$

respectively.

In addition, we have the only four-dimension algebra

$$\hat{A}_{4,1}: \quad e_1 = \tilde{g}_1,\ e_2 = g_5,\ e_3 = g_4,\ e_4 \tag{64}$$

which generates the following algebras of operators (25):

$$\begin{array}{l} < \mu D - \hat{e}_1,\ \nu D - \hat{e}_3,\ \hat{e}_2,\ \hat{e}_4 >,\ \ < \hat{e}_1,\ D + \hat{e}_3 + \nu t \hat{e}_4,\ \hat{e}_2,\ \hat{e}_4 >, \\ < D + \hat{e}_1 + \nu t \hat{e}_2,\ \hat{e}_2,\ \hat{e}_3,\ \hat{e}_4 > . \end{array} \tag{65}$$

Thus we have specified all low dimension algebras $\mathcal{A}$ which can be admitted by equations (2) with a diagonal (but not unit) matrix A.

The case $a \neq 1$ appears to be much more complicated. The related matrices g are of the most general form (45) and defined up to the general equivalence transformation (46), (47). In other words there are seven non-equivalent matrices (45), including g_1, g_2 (49) g_5 (54) and also the following matrices

$$\begin{array}{l} g_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mu & -1 \\ 0 & 1 & \mu \end{pmatrix},\quad g_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \\ \\ g_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\quad g_9 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{array} \tag{66}$$

In addition, we have fifteen two-dimension algebras of matrices (45),

$$\begin{array}{l} A_{2,1} = \{\tilde{g}_1, g_4\},\ A_{2,2} = \{\tilde{g}_1, \tilde{g}_3\},\ A_{2,3} = \{\tilde{g}_3, g_5\},\ A_{2,7} = \{g_7, g_8\}, \\ A_{2,8} = \{\tilde{g}_3, g_8\},\ A_{2,9} = \{\tilde{g}_3, g_9\},\ A_{2,10} = \{g_1', g_6\}, \end{array} \tag{67}$$

$$\begin{array}{l} A_{2,4} = \{g_1, g_5\},\ A_{2,5} = \{g_1', g_3\},\ A_{2,6} = \{g_2, \tilde{g}_3\},\ A_{2,11} = \{g_1|_{\lambda \neq 1}, g_8\}, \\ A_{2,12} = \{g_{11}, -g_8\},\ A_{2,13} = \{g_9, g''{}_1\},\ A_{2,14} = \{g_4, g_8\},\ A_{2,15} = \{g_7, \tilde{g}_3\} \end{array} \tag{68}$$

where

$$g_{10} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ g_1'' = g_1|_{\lambda=2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Algebras (67) are Abelian while algebras (68) are characterized by relations (55). The related algebras $\mathcal{A}$ are given by relations (56) for e_1, e_2 belonging to algebras (67); by relations (57) for e_1, e_2 belonging to algebras (67) and (68) and by relations (58) for e_1, e_2 belonging to algebras (68).

Three-dimension algebras are $A_{3,1} - A_{3,3}$ given by relations (60), (61) (where tildes should be omitted) and also $A_{3,4} - A_{3,11}$ given below:

$$\begin{array}{ll}
A_{3,4}: & e_1 = g_5,\ e_2 = g_6,\ e_3 = \tilde{g}_3, \\
A_{3,5}: & e_1 = g_8,\ e_2 = g_1',\ e_3 = \tilde{g}_3, \\
A_{3,6}: & e_1 = \tilde{g}_3,\ e_2 = g_8,\ e_3 = g_9, \\
A_{3,7}: & e_1 = \tilde{g}_3,\ e_2 = g_5,\ e_3 = g_7, \\
A_{3,8}: & e_1 = g_1,\ e_2 = g_8,\ e_3 = \tilde{g}_3, \\
A_{3,9}: & e_1 = g_4,\ e_2 = g_8,\ e_3 = \tilde{g}_3, \\
A_{3,10}: & e_1 = g_2,\ e_2 = g_8,\ e_3 = -\tilde{g}_3, \\
A_{3,11}: & e_1 = \tilde{g}_1,\ e_2 = -g_8,\ e_3 = \tilde{g}_4.
\end{array}$$

Algebras $(A_{3,8}, A_{3,11})$ and $A_{3,9}$ are isomorphic to $A_{3,1}$ and $A_{3,3}$ respectively. The related algebras $\mathcal{A}$ are given by relations (62) and (63) correspondingly.

Algebras $A_{3,6}$ and $A_{3,10}$ are characterized by the following commutation relations

$$[e_2, e_3] = e_1 \tag{69}$$

(the remaining commutators are equal to zero); non-zero commutators for basis elements of $A_{3,7}$ are given below:

$$[e_1, e_2] = e_2, \quad [e_1, e_3] = e_2 + e_3. \tag{70}$$

Using (69) and (70) we come to the following related three-dimension algebras $\mathcal{A}$ generated by $A_{3,6}$ and $A_{3,10}$:

$$\begin{array}{l}
< \mu D - 2\hat{e}_2,\ \nu D - 2\hat{e}_3,\ e_1 >,\ < e_1,\ D + 2e_\alpha + 2\nu t e_1,\ e_{\alpha'} >, \\
< e^{\nu t + \omega\cdot x} e_1,\ e^{\nu t + \omega\cdot x} e_\alpha,\ e_\alpha' >
\end{array} \tag{71}$$

and algebras (72) generated by $A_{3,7}$:

$$< \mu D - 2e_1,\ e_2,\ e_3 >,\ < e_1,\ e^{\nu t + \omega\cdot x} e_2, \quad e^{\nu t + \omega\cdot x} e_3 > . \tag{72}$$

Finally, four-dimension algebras of matrices (46) are $A_{4,1}$ given by equations (64) and also $A_{4,2}$–$A_{4,5}$ given below:

$$\begin{array}{ll}
A_{4,2}: & e_1 = g_1',\ e_2 = g_6,\ e_3 = \tilde{g}_3,\ e_4 = g_5; \\
A_{4,3}: & e_1 = \tilde{g}_3,\ e_2 = g_5,\ e_3 = g_1',\ e_4 = g_8; \\
A_{4,4}: & e_1 = g_1,\ e_2 = g_4,\ e_3 = g_8,\ e_4 = g_3; \\
A_{4,5}: & e_1 = g_4,\ e_2 = g_8,\ e_3 = g_5,\ e_4 = g_3.
\end{array}$$

We do not present the related algebras $\mathcal{A}$ because all possible non-linearities f^1 and f^2 will be fixed asking for invariance of equation (2) which respect to transformations generated by three-dimensional algebras.

8 Classification results

Using found algebras and solving the related classifying equations (26) we easily complete the group classification of equations (2). Namely, we solve the equations (26) with their known coefficient C^{ab} and B^a which are defined comparing (25) with the found realizations of algebras $\mathcal{A}$.

We consider consequently all one- two- and three-dimensional algebras $\mathcal{A}$ and find the related non-linearities. Then we control the cases when equation (2) admit extending symmetries G_μ, $\widehat{G}_\mu$ and K (17), i.e., when the found functions f^1 and f^2 satisfy conditions (27), (28) and (29) respectively.

We will not reproduce here the related routine calculations but present the results of group classification in Tables 2-8.

In the last columns of Tables 2–5 the additional equivalence transformations are presented. In Tables 2-8 the symbols D, G_μ, $\hat{G}_\nu$ and K denote generators (17), $\tilde{K} = K + \frac{1}{\lambda-1}\left[t\left(pu\partial_u + (2-\lambda)v\partial_v\right) + u\partial_v\right]$, ψ_μ is an arbitrary solution of the linear heat equation $\partial_t\psi_\mu - \Delta\psi_\mu = \mu\psi_\mu$, $\tilde{\psi}_\mu$ satisfies $\partial_t\tilde{\psi}_\mu - a\Delta\tilde{\psi}_\mu = \mu\tilde{\psi}_\mu$, $\varepsilon = \pm 1$ and $\Psi(x)$, $\Psi_\nu(x)$ have the same meaning as in Table 1. The items marked by asterisk are related to the unit diffusion matrix only.

Table 2
Non-linearities with arbitrary functions and extendible symmetries

No	Nonlinear terms	Arguments of F_a	Symmetries	Additional symmetries	AET
1.	$f^1 = u^{\nu+1}F_1$, $f^2 = u^{\nu+\mu}F_2$	$\frac{v}{u^\mu}$	$\nu D - u\partial_u - \mu v\partial_v$	G_α if $\nu = 0$, $\mu = a \neq 0$	1 if $\nu = 0$, $\rho = \mu\omega$
2.	$f^1 = u(F_1 + \nu\ln u)$, $\nu \neq 0$ $f^2 = v(F_2 + \nu\mu\ln u)$	$\frac{v}{u^\mu}$	$e^{\nu t}\left(u\partial_u + \mu v\partial_v\right)$	$\widehat{G}_\alpha$ if $\mu = a \neq 0$	
3*.	$f^1 = e^{\nu\frac{v}{u}}F_1u$, $f^2 = e^{\nu\frac{v}{u}}\left(F_1v + F_2\right)$	u	$\nu D - u\partial_v$	$v\partial_v$ if $\nu = 0$, $F_2 = 0$	6
				$\psi_0\partial_v$, $D + v\partial_v$, if $F_1 = 0$, $\nu = 0$	3,6
4*.	$f^1 = u(F_1 - \nu)$, $f^2 = F_1v + F_2$	u	$e^{\nu t}u\partial_v$	$\psi_\nu\partial_v$ if $F_1 = \nu$	3 if $\nu = 0$

Table 3
Non-linearities with arbitrary functions and non-extendible symmetries

No	Nonlinear terms	Arguments of $F_1,\ F_2$	Symmetries and AET (13) [in square brackets]
1*.	$f^1 = uF_1 + \nu v,$ $f^2 = \nu \frac{v}{u}(u+v)$ $+uF_2 + vF_1$	$ue^{-\frac{v}{u}}$	$e^{\nu t}\left(u\partial_v + u\partial_u + v\partial_v\right),$ [AET 1, $\rho = \omega$]
2*.	$f^1 = u^{\nu+1}F_{1,}$ $f^2 = u^{\nu}\left(F_1 v + F_2 u\right),$	$ue^{-\frac{v}{u}}$	$\nu D - u\partial_v - u\partial_u - v\partial_v,$ [AET 1, $\rho = \omega$]
3*.	$f^1 = uF_1 + vF_2$ $+\nu z\left(\mu u - v\right),$ $f^2 = vF_1 - uF_2$ $+\nu z\left(\mu v + u\right),\ \nu \neq 0$	$Re^{-\mu z}$ where $R = (u^2+v^2)^{\frac{1}{2}},$ $z = \tan^{-1}\left(\frac{v}{u}\right)$	$e^{\nu t}\left(\mu R\partial_R + \partial_z\right),$ [AET 15 if $\mu = 0$]
4.	$f^1 = v^{\nu}F_1,\ f^2 = v^{\nu+1}F_2$	$u - \ln v$	$\nu D - v\partial_v - \partial_u$ [AET 4 if $\nu = 0$]
5.	$f^1 = F_1 + \nu u,\ \nu \neq 0,$ $f^2 = F_2 v + \nu uv$	$u - \ln v$	$e^{\nu t}\left(v\partial_v + \partial_u\right)$
6*.	$f^1 = e^{\nu u}F_1,$ $f^2 = e^{\nu u}(F_2 + F_1 u)$	$2v - u^2$	$\nu D - u\partial_v - \partial_u$
7*.	$f^1 = \nu u + F_1,$ $f_2 = \nu u^2 + F_1 u + F_2$	$2v - u^2$	$e^{\nu t}\left(u\partial_v + \partial_u\right)$
8.	$f^1 = 0, \quad f^2 = F$	v	$\psi_0\partial_u, \quad u\partial_u$ [AET 2; $1, \rho = 0$]
9.	$f^1 = F_1, \quad f^2 = F_2 + \nu v$	u	$\tilde{\psi}_\nu \partial_v$
10.	$f^1 = F_1 + (\nu - \mu)u,$ $f^2 = F_2 + (\nu - a\mu)v$	$v - u$	$e^{\nu t}\Psi_\mu(x)\left(\partial_u + \partial_v\right)$
11.	$f^1 = e^{\nu v}F_1,\ f^2 = e^{\nu v}F_2$	$\mu v - u$	$\nu D - \mu\partial_u - \partial_v$
12*.	$f^1 = e^{\nu z}\left(F_1 v + F_2 u\right),$ $f^2 = e^{\nu z}\left(F_2 v - F_1 u\right)$	$R\,e^{-\mu z}$	$\nu D - \mu\left(u\partial_u + v\partial_v\right)$ $-\left(u\partial_v - v\partial_u\right)$
13*.	$f^1 = \alpha u + \mu,$ $f_2 = \nu v + F, \quad \alpha\mu = 0$	u	$\psi_\nu \partial_v,$ $e^{(\nu-\alpha)t}\left(u - \mu t\right)\partial_v$
14*.	$f^1 = u^2,$ $f_2 = (u+\nu)v + F,$	u	$e^{\nu t}u\partial_v,$ $e^{\nu t}\left(\partial_v + tu\partial_v\right)$
15*.	$f^1 = \left(u^2 - 1\right),$ $f_2 = \left(u+\nu\right)v + F$	u	$e^{(\nu+1)t}\left(u\partial_v + \partial_v\right),$ $e^{(\nu-1)t}\left(u\partial_v - \partial_v\right)$
16*.	$f^1 = \left(u^2 + 1\right),$ $f_2 = \left(u+\nu\right)v + F$	u	$e^{\nu t}\left(\cos tu\partial_v - \sin t\partial_v\right),$ $e^{\nu t}\left(\sin tu\partial_v + \cos t\partial_v\right)$

Table 4. Non-linearities with arbitrary parameters and extendible symmetries

No	Nonlinear terms	Main symmetries	Additional symmetries	AET (13)
1.	$f^1=\lambda u^{\nu+1}v^{\mu}$, $f^2=\sigma u^{\nu}v^{\mu+1}$	$\mu D-v\partial_v$, $\nu D-u\partial_u$	G_μ if $\nu=-a\mu\neq 0$, & K if $a\neq 1$, $\mu=\frac{4}{m(1-a)}$;	$1, \nu\omega+\mu\rho=0$
			$\psi_0\partial_u$ if $\sigma=0, \nu=-1$ & G_α if $\mu=\frac{1}{a}$, $a\neq 0$, & K if $a=\frac{4}{4+m}$;	$2;\ 1, \nu\omega+\mu\rho=0$
			$\psi_0\partial_u, v\partial_u$ if $\sigma=0$, $\nu=-1$, $a=1$;	$2;\ 1, \nu\omega+\mu\rho=0$
			G_α, K if $\sigma=1, \lambda\neq 1$, $\mu=-\nu=a=1$;	$1, \nu\omega+\mu\rho=0$
			$u\partial_v$ if $\mu=0$, $\lambda=\sigma, a=1$ & $\partial_v+tu\partial_v$ if $\nu=1$	$1, \nu\omega+\mu\rho=0$
2.	$f^1=\lambda u^{\nu+1}$, $f^2=\sigma u^{\nu+\mu}$	$\nu D-u\partial_u-\mu v\partial_v$, $\tilde{\psi}_0\partial_v$	$(u-\lambda t)\,\partial_v$ if $-\nu=a=1$	$3, 6$
			G_α if $\nu=0$, $\mu=a\neq 0$;	$3, 6$
			$e^{-\lambda t}u\partial_v$ if $\nu=0$, $a=1$, & $e^{\lambda t}\,(u\partial_v+\lambda\partial_u)$ if $\mu=2$, & $2D-u\partial_u$ if $\lambda=0$	$3;\ 1, \rho=\omega$
3.	$f^1=\lambda e^{\nu u}$, $f^2=\sigma e^{(\nu+1)u}$	$\nu D-v\partial_v-\partial_u$, $\tilde{\psi}_0\partial_v$	$v\partial_v$ if $\sigma=0$	$3;\ 1, \omega=0$
			$(u-\lambda t)\,\partial_v$ if $\nu=0$, $a=1$	$3, 4$
4.	$f^1=\lambda e^{v}$, $f^2=\sigma e^{v}$	$D-\partial_v$, $\psi_0\partial_u$	$u\partial_u+\partial_v$ if $\sigma=0$ & $v\partial_u$ if $a=1$	$2, 5$
5*.	$f^1=\lambda u^{\nu+1}e^{\mu\frac{v}{u}}$, $f^2=e^{\mu\frac{v}{u}}(\lambda v+\sigma u)u^{\nu}$	$\mu D-u\partial_v$, $\nu D-u\partial_u-v\partial_v$	G_α if $\nu=0$	$1, \rho=\omega$

Table 4. Continued

6*.	$f^1 = e^{\mu z}R^\nu(\lambda u - \sigma v)$, $f^2 = e^{\mu z}R^\nu(\lambda v + \sigma u)$	$\nu D - u\partial_u - v\partial_v$, $\mu D - u\partial_v + v\partial_u$	G_α if $\nu = 0$	1, $\rho = \omega$
7*.	$f^1 = \nu u^{\mu+1}$, $f^2 = \sigma u^\mu(v - \ln u)$, $\mu \neq 0$,	$\mu D - u\partial_u - 2\partial_v$, $u\partial_v$	$\partial_v + tu\partial_v$ if $\mu = 1$	6
8.	$f^1 = \lambda$, $f^2 = \varepsilon \ln u$	$D + u\partial_u + v\partial_v + \varepsilon t\partial_v$, $\tilde{\psi}_0\partial_v$	For $a \neq 1$: $u\partial_u + \varepsilon t\partial_v$ if $\lambda = 0$;	$3, 7$
			For $a = 1$: $(u - \lambda t)\,\partial_v$; & $u\partial_u + \varepsilon t\partial_v$ if $\lambda = 0$	$8, 9$
9.	$f^1 = 0$, $f^2 = \varepsilon v + \ln u$	$\mu u\partial_u - \varepsilon\partial_v$, $\tilde{\psi}_\varepsilon\partial_v$	$e^{\varepsilon t}u\partial_v$ if $a = 1$	10, $\kappa = \varepsilon$
10.	$f^1 = \lambda u \ln u$, $f^2 = \nu v + \ln u$	$\tilde{\psi}_\nu\partial_v$	$e^{\nu t}\,(u\partial_u + t\partial_v)$ if $\nu = \lambda$;	10, $\kappa = \varepsilon$
			$e^{\lambda t}\,((\lambda - \nu)u\partial_u + \partial_v)$ if $\nu \neq \lambda$	10, $\kappa = \varepsilon$

Table 5. Non-linearities with arbitrary parameters and non-extendible symmetries

	Nonlinear terms	Symmetries	AET (13)
1.	$f^1 = \lambda\,(u + v)^{\nu+1}$, $f^1 = \mu\,(u + v)^{\nu+1}$, $a \neq 1$	$\nu D - u\partial_u - v\partial_v$, $\Psi_0(x)\,(\partial_u - \partial_v)$	12
2*.	$f^1 = \lambda u^{\nu+1}$, $f^2 = u^\nu\,(\lambda u_2 + \mu u^\sigma)$, $\nu + \sigma \neq 1$, $\mu \neq 0$	$\nu D - u\partial_u - \sigma v\partial_v$, $u\partial_v$	6
3.	$f^1 = \lambda e^{(u+v)}$, $f^2 = \sigma e^{(u+v)}$, $a \neq 1$	$D - \partial_v$, $\Psi_0(x)\,(\partial_u - \partial_v)$	12
4.	$f^1 = \lambda v^\nu e^u$, $f^2 = \sigma v^{\nu+1}e^u$	$D - \partial_u$, $v\partial_v - \nu\partial_u$	13, $\nu \neq 0$
5*.	$f^1 = \lambda e^u$, $f^2 = \sigma u e^u$	$D - \partial_u - u\partial_v$, $\psi_0\partial_v$	3
6.	$f^1 = \varepsilon e^u$, $\varepsilon = \pm 1$, $f^2 = \lambda u$	$D + u\partial_v - \partial_u - \lambda t\partial_v$, $\psi_0\partial_v$	3
7*.	$f^1 = \nu e^{\lambda\left(2v - u^2\right)}$, $f^2 = (\nu u + \mu)\,e^{\lambda\left(2v - u^2\right)}$	$\lambda D - \partial_v$, $\partial_u + u\partial_v$	14
8.	$f^1 = \lambda u^{\nu+1}$, $f^2 = \ln u$, $\lambda(\nu + 1) \neq 0$	$\nu\,(D + 2v\partial_v) - 2u\partial_u - 2t\partial_v$, $\tilde{\psi}_0\partial_v$	3

Table 5. Continued

9*.	$f^1=\lambda\ln(2v-u^2)$, $f^2=\sigma(2v-u^2)$ $+\lambda u\ln(2v-u^2)$	$D+u\partial_u+2v\partial_v$ $+2\lambda t\left(\partial_u+u\partial_v\right)$, $\partial_u+u\partial_v$	14
10.	$f^1=\ln\left(u+v\right)$, $f^2=\nu\ln\left(u+v\right)$, $a\neq 1$	$\Psi_0(x)\left(\partial_u-\partial_v\right)$, $(a-1)\left(D+u\partial_u\right.$ $\left.+v\partial_v\right)+\left((a-\nu)t\right.$ $\left.+\frac{1-\nu}{2m}x^2\right)\left(\partial_u-\partial_v\right)$	12
11*.	$f^1=0$, $f^2=\varepsilon(v-u\ln u)$	$u\partial_u+v\partial_v+u\partial_v$, $\tilde{\psi}_\varepsilon\partial_v$	11
12*.	$f^1=\lambda u^{\nu+1}$, $f^2=\lambda u^{\nu+1}\ln u$	$\nu D-(u\partial_u+v\partial_v$ $+u\partial_v)$, $\psi_0\partial_v$	3
13*.	$f^1=\lambda u^{\nu+1},\ \lambda\neq 0$, $f^2=\lambda u^\nu v\ +u\ln u$	$\nu D-u\partial_u-tu\partial_v$ $-(1-\nu)v\partial_v,\ u\partial_v$	6
14*.	$f^1=\lambda\left(2v-u^2\right)^{\nu+\frac{1}{2}}$, $f^2=\lambda u\left(2v-u^2\right)^{\nu+\frac{1}{2}}$ $+\mu\left(2v-u^2\right)^{\nu+1}$	$2\nu D-u\partial_u-2v\partial_v$, $\partial_u+u\partial_v$	14; 1, $\rho=2\omega$
15.	$f^1=(\mu-\nu)u\ln u+uv$, $f^2=-\nu^2\ln u$ $+(\mu+\nu)v$	$X=e^{\mu t}\left(u\partial_u+\nu\partial_v\right)$, $tX+e^{\mu t}\partial_v$	10, κ $=\mu-\nu$ if $\mu=0$
16.	$f^1=(\mu-\nu)u\ln u+uv$, $f^2=(1-\nu^2)\ln u$ $+(\mu+\nu)v$	$e^{\lambda_+t}\left(u\partial_u+\lambda_-\partial_v\right)$, $e^{\lambda_-t}\left(u\partial_u-\lambda_+\partial_v\right)$, $\lambda_\pm=\mu\pm 1$	10, κ $=\mu-\nu$ if $\mu=\pm1$
17.	$f^1=(\mu-\nu)u\ln u+uv$, $f^2=(\mu+\nu)v$ $-\left(1+\nu^2\right)\ln u$	$e^{\mu t}\left[\cos tu\partial_u\right.$ $\left.+\nu\left(\cos t-\sin t\right)\partial_v\right]$, $e^{\mu t}\left[\sin tu\partial_u\right.$ $\left.+\nu(\sin t+\cos t)\partial_v\right]$	
18*.	$f^1=2\varepsilon v-u^2$, $f^2=-\frac{\mu^2}{2}u$ $+(\mu+u)\left(2\varepsilon v-u^2\right)$	$X_1=e^{\mu t}\left(2\varepsilon\partial_u+2u\partial_v\right.$ $\left.+\mu\partial_v\right)$, $2tX_1+e^{\mu t}\partial_v$	14, if $\mu=0$
19*.	$f^1=2\varepsilon v-u^2$, $f^2=\frac{1-\mu^2}{2}u$ $+(\mu+u)\left(2\varepsilon v-u^2\right)$	$X^\pm=e^{\mu\pm1}\left(2\partial_u\right.$ $\left.+2\varepsilon u\partial_v+(\mu\pm1)\partial_v\right)$	14, if $\mu^2=1$
20*.	$f^1=2v-\varepsilon u^2$, $f^2=-\frac{1+\mu^2}{2}u$ $+(\mu+\varepsilon u)\left(2v-\varepsilon u^2\right)$	$e^{\mu t}\left(2\cos t\left(\partial_u+\varepsilon u\partial_v\right)\right.$ $\left.+(2\mu\cos t-\sin t)\partial_v\right)$, $e^{\mu t}\left(2\sin t\left(\partial_u+\varepsilon u\partial_v\right)\right.$ $\left.+(2\mu\sin t+\cos t)\partial_v\right)$	

In the following tables $\delta=\frac{1}{4}(\mu-\nu)^2+\lambda\sigma$. All equations presented in Table 6 admit the additional equivalence transformations $u\to u\cos\omega t-v\sin\omega t$, $v\to v\cos\omega t+u\sin\omega t$.

Table 6. Symmetries of equations (2) with non-linearities
$f^1=(\mu u-\sigma v)\ln R+z(\lambda u-\nu v)$, $f^2=(\mu v+\sigma u)\ln R+z(\lambda v+\nu u)$

No	Conditions for coefficients	Main symmetries	Additional symmetries for $a\neq 0$
1^*	$\lambda=0,\ \mu=\nu$	$e^{\mu t}\partial_z,\ e^{\mu t}\left(R\partial_R+\sigma t\partial_z\right)$	$\widehat{G}_\alpha$ if $\sigma=0,\mu\neq 0$
2^*	$\lambda=0,\ \mu\neq\nu$	$e^{\nu t}\partial_z$, $e^{\mu t}\left(\sigma\partial_z+(\mu-\nu)R\partial_R\right)$	G_α if $\mu=\sigma=0$ $\widehat{G}_\alpha$ if $\mu\neq 0,\sigma=0$
3^*	$\delta=0,\ \lambda\neq 0,$ $\mu+\nu=2\omega_0$	$X_3=e^{\omega_0 t}(2\lambda R\partial_R+(\nu$ $-\mu)\partial_z)$, $2e^{\omega_0 t}\partial_z+tX_3$	$\widehat{G}_\alpha$ if $\mu=\nu\neq 0$, G_α if $\mu=\nu=0$
4^*	$\lambda\neq 0,\ \delta=1$ $\omega_\pm=\omega_0\pm 1$	$e^{\omega_+ t}\left(\lambda R\partial_R+(\omega_+-\mu)\partial_z\right)$, $e^{\omega_- t}\left(\lambda R\partial_R+(\omega_--\mu)\partial_z\right)$	$\widehat{G}_\alpha$ if $\omega_+=\mu\neq 0$ G_α if $\omega_+=\mu=0$
5^*	$\delta=-1$	$\exp(\omega_0 t)\left[2\lambda\cos tR\partial_R+((\nu-\mu)\cos t-\sin t)\partial_z\right]$, $\exp(\omega_0 t)\left[2\lambda\sin tR\partial_R+((\nu-\mu)\sin t+\cos t)\partial_z\right]$	none

Table 7. Symmetries of equations (2) with non-linearities
$f^1=\lambda v+\mu u\ln u$, $f^2=\lambda\frac{v^2}{u}+(\sigma u+\mu v)\ln u+\nu v+\alpha u$

No	Conditions for coefficients	Main symmetries	Additional symmetries
1^*	$\lambda=0,\ \mu\neq\nu,$ $\alpha\nu=0$	$e^{\nu t}u\partial_v$, $e^{\mu t}\left((\mu-\nu)R\partial_R+\sigma u\partial_v\right)$	$v\partial_v+\alpha tu\partial_v,\widehat{G}_a$ if $\mu\neq 0,\sigma=0$
2^*	$\sigma=0,$ $\lambda\neq 0,$ $\mu\neq\nu$	$e^{\nu t}\left(\lambda R\partial_R+(\mu-\nu)u\partial_v\right)$ $e^{\mu t}R\partial_R$	G_a if $\mu=\alpha=0$ $\hat{G}_a$ if $\mu\neq 0$
3^*	$\delta=0$ $\mu+\nu=2\omega_0$	$X_4=e^{\omega_0 t}(2\lambda R\partial_R$ $+(\nu-\mu)u\partial_v)$, $2e^{\omega_0 t}u\partial_v+tX_4$	G_a if $\mu=\nu=0$, $\lambda\neq 0$, & $D+u\partial_u$ if $\alpha=0$ $\hat{G}_a$ if $\mu=\nu\neq 0$, $\lambda\neq 0$ $R\partial_R+\sigma tu\partial_v,\psi_0\partial_u$, $D+v\partial_v$ if $\sigma\neq 0$ $\lambda=\mu=\nu=\alpha=0$
4^*	$\lambda\neq 0,$ $\delta=1,$ $\omega_\pm=\omega_0\pm 1$	$e^{\omega_+ t}\left(\lambda R\partial_R+(\omega_+-\mu)u\partial_v\right)$, $e^{\omega_- t}\left(\lambda R\partial_R+(\omega_--\mu)u\partial_v\right)$	G_a if $\omega_+=\mu=0$ $\hat{G}_\alpha$ if $\omega_+=\mu\neq 0$
5^*	$\delta=-1,$	$e^{\omega_0 t}[2\lambda\cos tR\partial_R$ $+((\nu-\mu)\cos t-\sin t)u\partial_v]$, $e^{\omega_0 t}[2\lambda\sin tR\partial_R$ $+((\nu-\mu)\sin t+\cos t)u\partial_v]$	none

Table 8. Symmetries of equations (2) with non-linearities
$f^1 = u\,(\mu \ln u + \lambda \ln v)\,,\ \ f^2 = v\,(\nu \ln v + \sigma \ln u)$

No	Conditions for coefficients	Main symmetries	Additional symmetries
1	$\lambda = 0,\ \mu = \nu$	$e^{\mu t} v\partial_v$, $e^{\mu t}\,(u\partial_u + \sigma t v\partial_v)$	$\psi_0\partial_u$ if $\mu = 0, \sigma \neq 0$
			$\hat{G}_\mu$ if $a \neq 0,\ \sigma = 0,\ \mu \neq 0$
2	$\lambda = 0$, $\mu \neq \nu$	$e^{\mu t}\,((\mu - \nu)\,u\partial_u + \sigma v\partial_v)\,,$ $e^{\nu t} v\partial_v$	$\widehat{G}_\alpha$ if $\mu \neq 0$, $a(\mu - \nu) = \sigma,\ a \neq 0$
			G_α if $\sigma = -\nu a, \mu = 0, a \neq 0$
			$\psi_0\partial_v$ if $\sigma = \nu = 0$;
			$u\partial_v,\ \widehat{G}_\alpha$ if $a = 1$, $\nu = 0,\ \mu = \sigma \neq 0$
3	$\delta = 0$, $\mu + \nu = 2\omega_0$, $\lambda\sigma \neq 0$	$X_2 = e^{\omega_0 t}\,(2\lambda u\partial_u + (\nu - \mu)\,v\partial_v)\,,$ $e^{\omega_0 t} 2v\partial_v + tX_2$	$\hat{G}_\alpha$ if $\nu \neq -\mu$, $2\lambda a = \nu - \mu,\ a \neq 0$
			G_α if $\mu \neq 0$, $\lambda a = \nu = -\mu,\ a \neq 0$
4	$\lambda\sigma \neq 0$, $\delta = 1$, $\omega_\pm = \omega_0 \pm 1$	$e^{\omega_+ t}\,(\lambda u\partial_u + (\omega_+ - \mu)\,v\partial_v)\,,$ $e^{\omega_- t}\,(\lambda u\partial_u + (\omega_- - \mu)\,v\partial_v)$	$\hat{G}_\alpha$ if $\omega_+ \neq 0$, $a\lambda = \omega_+ - \mu$
			G_α if $\mu = -a\lambda$, $\omega_+ = 0$,
5	$\delta = -1$	$e^{\omega_0 t}\,(2\lambda \cos t u\partial_u - ((\mu - \omega_0)\cos t + \sin t)\,v\partial_v)\,,$ $e^{\omega_0 t}\,(2\lambda \sin t u\partial_u + ((\omega_0 - \mu)\sin t + \cos t)\,v\partial_v)$	none

Equations (2) with the nonlinearities present in Table 8 admit equivalence transformation 1 from the list (13) provided $\mu\sigma = \lambda\nu$. The related parameters ρ and ω should satisfy $\mu\omega + \lambda\rho = 0$. In addition, the equations corresponding to the last version enumerated in Item 2 admit additional equivalence transformation Number 6 which is given by formula (13).

9 Discussion

Thus we carry out the group classification of systems of coupled reaction-diffusion equations (2) with a diagonal diffusion matrix. The classification results are present in Tables 2-8. In addition, symmetries of equation (6) with singular diffusion matrix and additional first derivative terms are presented in Table 1.

Classification results given in Tables 2-5 and 8 (those ones which are not marked by asterisk) are valid for both invertible and singular diffusion matrices.

However, the Tables do not include all classification results for equations (2) with $a = 0$. To complete the list of symmetries for the case of singular diffusion matrix it is necessary to add "mirror" versions which can be obtained from Items 1,2 of Table 2, Items 4, 5, 8, 9, 11 of Table 3, Items 1-4, 8-10 of Table 4, Items 4, 6, 8, 15-17 of Table 5 and Items 1-5 of Table 8 by the simultaneous change

$$u \to v,\ v \to u,\ f^1 \to f^2,\ f^2 \to f^1,\ \psi_\nu \to \Psi_\nu,\ \tilde{\psi}_\nu \to \psi_\nu,\ \Psi_0(x) \to 1$$

The list of non-equivalent systems (2) appears to be very extended, especially in the case of unit diffusion matrix. Equations (2) with invertible and non-unit diffusion matrix A have relative short list of different symmetries. More exactly, if matrix matrix A has type 1, equation (12), then there exist 15 non-equivalent classes of equations (2) defined up to arbitrary functions and 46 classes of such equations defined up to arbitrary parameters. The related extensions of the basic symmetries (24) have dimensions from 1 up to 3 and include neither Galilei generators G_α nor conformal generators K.

For the case when matrix A is of type 2, equation (12), we indicate 10 classes of equations depending on arbitrary functions and 41 class of equations defined up to arbitrary or fixed parameters. Among them there are 7 Galilei invariant systems and two equations invariant w.r.t. extended Galilei algebra spanned on $P_\mu, J_{\mu\nu}$ (24) dilatation operator and also generators G_α, K (17). These equations are indicated in Table 4, Item 1 and have the following form

$$\begin{aligned} u_t - \Delta u &= \lambda u \left(u^a v^{-1}\right)^{\frac{4}{m(a-1)}}, \\ v_t - a\Delta v &= \sigma v \left(u^a v^{-1}\right)^{\frac{4}{m(a-1)}} \end{aligned}$$

and

$$u_t - \Delta u = \lambda v^{\frac{4+m}{4}},\ v_t - a\Delta v = 0.$$

Finally, if the diffusion matrix is the unit one then we have 115 non-equivalent classes of equations, among them 24 including arbitrary functions and 13 admitting Galilei generators. There is the only equation admitting extended Galilei algebra, the related non-linearities are given in Table 4, Item 1 for $\sigma = \mu = -\nu = 1$.

Let us compare our results with those of [2] and [5], [6].

Paper [2] was apparently the first work were the problem of group classification of equations (2) with a diagonal diffusion matrix was formulated and partially solved. Unfortunately, the classification results presented in [2] are incomplete and in many points incorrect. Thus, all cases enumerated above in Table 8, Items 1, 2 of Table 2, Items 4, 5, 8-11, of Table 4, Items 1, 3, 10, 15-17 of Table 6, were overlooked, symmetries of equations with non-linearities given in Items 1 and 2 of Table 4 were presented incompletely, etc.

In papers [5], [6] Lie symmetries of the same equations and also of systems of diffusion equations with the unit diffusion matrix were classified. The results present in those papers are much more advanced then the pioneer Davidov ones, nevertheless they are still incomplete. In particular, the cases indicated above in Items 3 and 4 of Table 2; Items 15-17 of Table 3; Items 11 and 20 of Table 5, Items 1, 2 of Table 6 were not indicated in [6]. Equivalence relations are neither clearly specified nor systematically used, which results in appearance of great many of unnecessary arbitrary parameters in classification results present in [5], [6]. Moreover, many of equations presented in [6] as non-equivalent ones, in fact are equivalent one to another. It is possible to indicate at least ten examples of such cases, half of them in frames of equivalence relations (7) declared in the mentioned paper. For instance, all versions 14, 15, 18 and 20 from Table 4 present in [6] are equivalent one to another.

The other examples of equivalent equations treated in [6] as non-equivalent will not be enumerated here in as much as we believe that all non-equivalent equations (2) with different symmetries are present in Tables 2-8.

Consider examples of well known reaction diffusion equations which appear to be particular subjects of our analysis.

The Jackiw-Teitelboim model of two-dimension gravity with the non-relativistic gauge [7] admits the equivalence transformation 1 (13) for $\rho = -\omega$. Choosing $\rho = 2k$ we transform equation (3) to the form (1) where $a = -1$, $f^1 = -2u^2v$ and $f^2 = 2v^2u$. The symmetries corresponding to these non-linearities are given in the first line of Table 5. Symmetries of equations (3) were investigated in paper [8] whose results are in accordance with our analysis. We notice also that generalized equation (3) with two spatial variables admits additional conformal symmetry generated by operator K (17).

The primitive predator-prey system (4) is a particular case of equation (1) with the non-linearities given in the first line of Table 2 where however $\mu = \nu = 1, F_1 = -F_2 = \frac{u}{v}$. In addition to the basic symmetries $< \frac{\partial}{\partial t}, \frac{\partial}{\partial x} >$ this equation admits the (main) symmetry:

$$X = \left(D - 2u\frac{\partial}{\partial u} - 2v\frac{\partial}{\partial v} \right).$$

The $\lambda - \omega$ reaction-diffusion system (5) and its symmetries was studied in paper [12]. Again we recognize that this system is a particular case of (1) with non-linearities given in Item 12* of Table 3 with $\mu = \nu = 0$. Hence it admits the five dimensional Lie algebra generated by basic symmetries (24) with $\mu, \nu = 1, 2$ and:

$$X = \left(u\frac{\partial}{\partial v} - v\frac{\partial}{\partial u} \right) \tag{73}$$

which is in accordance with results of paper [12] for arbitrary functions λ and ω. Moreover, using Table 4, Item 6* we find that for the cases when

$$\lambda(R) = \tilde{\lambda} R^{\nu}, \quad \omega = \sigma R^{\nu} \tag{74}$$

equation (5) admits additional symmetry with respect to scaling transformations generated by the operator:

$$X = \left(u\frac{\partial}{\partial v} - v\frac{\partial}{\partial u}\right) + \nu D. \tag{75}$$

The other extensions of the basic symmetries correspond to the case when $\lambda(R) = \mu \ln(R), \omega(R) = \sigma \ln(R)$, the related additional symmetries are given in Table 7 where $\nu = \lambda = 0$.

Detailed analysis of symmetries of $\lambda - \omega$ reaction-diffusion systems is given in paper [6].

Thus we present group classification of reaction-diffusion systems with diagonal diffusion matrix. Such systems with the square diffusion matrix has been classified in [1] while the case of triangular diffusion matrix will be a subject of the following paper.

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