1.1. Heat Equation \( \frac{\partial w}{\partial t} = \alpha \frac{\partial^2 w}{\partial x^2} \)

1.1-1. Particular solutions of the heat (diffusion) equation:

\[
\begin{align*}
  w(x) &= Ax + B, \\
  w(x, t) &= A(x^2 + 2at) + B, \\
  w(x, t) &= A(x^3 + 6atx) + B, \\
  w(x, t) &= A(x^3 + 12atx^2 + 12a^2t^2) + B, \\
  w(x, t) &= x^{2n} + \sum_{k=1}^{n} \frac{(2n)(2n-1)\ldots(2n-2k+1)}{k!} (at)^k x^{2n-2k}, \\
  w(x, t) &= x^{2n+1} + \sum_{k=1}^{n} \frac{(2n+1)(2n)\ldots(2n-2k+2)}{k!} (at)^k x^{2n-2k+1}, \\
  w(x, t) &= A \exp(a\mu^2 t \pm \mu x) + B, \\
  w(x, t) &= A \frac{1}{\sqrt{t}} \exp\left(-\frac{x^2}{4at}\right) + B, \\
  w(x, t) &= A \exp(-a\mu^2 t) \cos(\mu x + B) + C, \\
  w(x, t) &= A \exp(-\mu x) \cos(\mu x - 2\mu^2 t + B) + C, \\
  w(x, t) &= A \text{erf}\left(\frac{x}{2\sqrt{at}}\right) + B,
\end{align*}
\]

where \( A, B, C, \) and \( \mu \) are arbitrary constants, \( n \) is a positive integer, \( \text{erf} \equiv \frac{2}{\sqrt{\pi}} \int_0^z \exp(-\xi^2) d\xi \) is the error function (probability integral).

1.1-2. Formulas allowing the construction of particular solutions for the heat equation. Suppose \( w = w(x, t) \) is a solution of the heat equation. Then the functions

\[
\begin{align*}
  w_1 &= Aw(\pm \lambda x + C_1, \lambda^2 t + C_2) + B, \\
  w_2 &= A \exp(\lambda x + \alpha^2 t)w(x + 2at + C_1, t + C_2), \\
  w_3 &= \frac{A}{\sqrt{4(\delta + \beta t)}} \exp\left(-\frac{\beta x^2}{4(\delta + \beta t)}\right) w\left(\pm \frac{x}{\delta + \beta t}, \frac{\gamma + \lambda t}{\delta + \beta t}\right), \quad \lambda \delta - \beta \gamma = 1,
\end{align*}
\]

where \( A, B, C_1, C_2, \beta, \delta, \) and \( \lambda \) are arbitrary constants, are also solutions of this equation. The last formula with \( \beta = 1, \gamma = -1, \delta = \lambda = 0 \) was obtained with the Appell transformation.


For solutions of the Cauchy problem and various boundary value problems, see [nonhomogeneous heat equation] with \( \Phi(x, t) = 0 \).

1.1-4. Other types of heat equations.

See also related linear equations:
- [nonhomogeneous heat equation],
- [convective heat equation with a source],
- [heat equation with axial symmetry].
• nonhomogeneous heat equation with axial symmetry,
• heat equation with central symmetry,
• nonhomogeneous heat equation with central symmetry.

References