1.1. Heat Equation $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2}$

1.1-1. Particular solutions of the heat (diffusion) equation:

$$
\begin{aligned}
w(x) &= Ax + B,\\
w(x,t) &= A(x^2 + 2at) + B,\\
w(x,t) &= A(x^3 + 6atx) + B,\\
w(x,t) &= A(x^4 + 12atx^2 + 12a^2t^2) + B,\\
w(x,t) &= x^{2n} + \sum_{k=1}^{n} \frac{(2n)(2n-1)\ldots(2n-2k+1)}{k!}(at)^k x^{2n-2k},\\
w(x,t) &= x^{2n+1} + \sum_{k=1}^{n} \frac{(2n+1)(2n)\ldots(2n-2k+2)}{k!}(at)^k x^{2n-2k+1},\\
w(x,t) &= A\exp(a\mu^2 t \pm \mu x) + B,\\
w(x,t) &= A\frac{1}{\sqrt{t}}\exp\left(-\frac{x^2}{4at}\right) + B,\\
w(x,t) &= A\exp(-a\mu^2 t)\cos(\mu x + B) + C,\\
w(x,t) &= A\exp(-\mu x)\cos(\mu x - 2a\mu^2 t + B) + C,\\
w(x,t) &= A\,\mathrm{erf}\left(\frac{x}{2\sqrt{at}}\right) + B,
\end{aligned}
$$

where A, B, C, and μ are arbitrary constants, n is a positive integer, $\mathrm{erf}\, z \equiv \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^z \exp(-\xi^2)\,d\xi$ is the error function (probability integral).

1.1-2. Formulas allowing the construction of particular solutions for the heat equation.

Suppose $w = w(x,t)$ is a solution of the heat equation. Then the functions

$$
\begin{aligned}
w_1 &= Aw(\pm\lambda x + C_1,\ \lambda^2 t + C_2) + B,\\
w_2 &= A\exp(\lambda x + a\lambda^2 t)w(x + 2a\lambda t + C_1,\ t + C_2),\\
w_3 &= \frac{A}{\sqrt{|\delta + \beta t|}}\exp\left[-\frac{\beta x^2}{4a(\delta + \beta t)}\right] w\left(\pm\frac{x}{\delta + \beta t}, \frac{\gamma + \lambda t}{\delta + \beta t}\right), \qquad \lambda\delta - \beta\gamma = 1,
\end{aligned}
$$

where A, B, C_1, C_2, β, δ, and λ are arbitrary constants, are also solutions of this equation. The last formula with $\beta = 1$, $\gamma = -1$, $\delta = \lambda = 0$ was obtained with the Appell transformation.

1.1-3. Cauchy problem and boundary value problems for the heat equation.

For solutions of the Cauchy problem and various boundary value problems, see nonhomogeneous heat equation with $\Phi(x,t) \equiv 0$.

1.1-4. Other types of heat equations.

See also related linear equations:
- nonhomogeneous heat equation ,
- convective heat equation with a source ,
- heat equation with axial symmetry ,

- nonhomogeneous heat equation with axial symmetry ,
- heat equation with central symmetry ,
- nonhomogeneous heat equation with central symmetry .

References

Carslaw, H. S. and Jaeger, J. C., *Conduction of Heat in Solids*, Clarendon Press, Oxford, 1984.

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