1.4. Heat Equation with Axial Symmetry

\[ \frac{\partial w}{\partial t} = a \left( \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right) \]

This is the heat (diffusion) equation with axial symmetry, where \( r = \sqrt{x^2 + y^2} \) is the radial coordinate.

1.4-1. Particular solutions of the heat equation with axial symmetry:

\[
\begin{align*}
  w(r) & = A + B \ln r, \\
  w(r, t) & = A + B(r^2 + 4at), \\
  w(r, t) & = A + B(r^4 + 16atr^2 + 32a^2t^2), \\
  w(r, t) & = A + B \left( r^{2n} + \sum_{k=1}^{n} \frac{4^k [n(n-1) \ldots (n-k+1)]^2}{k!} (at)^k r^{2n-2k} \right), \\
  w(r, t) & = A + B(4at \ln r + r^2 \ln r - r^2), \\
  w(r, t) & = A + \frac{B}{t} \exp \left( -\frac{r^2}{4at} \right), \\
  w(r, t) & = A + B \exp(-at \ln r), \\
  w(r, t) & = A + B \exp(-at r^2), \\
  w(r, t) & = A + B \exp \left( -\frac{r^2 + \mu^2}{4t} \right) J_0 \left( \frac{\mu r}{2t} \right), \\
  w(r, t) & = A + B t \exp \left( -\frac{r^2 + \mu^2}{4t} \right) K_0 \left( \frac{\mu r}{2t} \right),
\end{align*}
\]

where \( A, B, \) and \( \mu \) are arbitrary constants, \( n \) is an arbitrary positive integer, \( J_0(z) \) and \( Y_0(z) \) are the Bessel functions, and \( I_0(z) \) and \( K_0(z) \) are the modified Bessel functions.

1.4-2. Formulas allowing the construction of particular solutions.

Suppose \( w = w(r, t) \) is a solution of the heat equation. Then the functions

\[
\begin{align*}
  w_1 & = A w(\pm \lambda r, \lambda^2 t + C) + B, \\
  w_2 & = \frac{A}{\delta + \beta t} \exp \left( -\frac{\beta r^2}{4a(\delta + \beta t)} \right) w \left( \pm \frac{r}{\delta + \beta t}, \frac{\gamma + \lambda t}{\delta + \beta t} \right),
\end{align*}
\]

where \( A, B, C, \beta, \delta, \) and \( \lambda \) are arbitrary constants, are also solutions of this equation. The second formula usually may be encountered with \( \beta = 1, \gamma = -1, \) and \( \delta = \lambda = 0.\)

1.4-3. Boundary value problems for the heat equation with axial symmetry.

For solutions of various boundary value problems, see [Subsection 1.5].

References
