1.5. Heat Equation of the Form \( \frac{\partial w}{\partial t} = \alpha \left( \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right) + \Phi(r, t) \)

**Nonhomogeneous heat (diffusion) equation with axial symmetry.**

1.5-1. Solutions of boundary value problems in terms of the Green’s function.

We consider boundary value problems for the nonhomogeneous heat equation with axial symmetry in domain \( 0 \leq r \leq R \) with the general initial condition

\[ w = f(r) \quad \text{at} \quad t = 0 \]

and various homogeneous boundary conditions (the solutions bounded at \( r = 0 \) are sought). The solution can be represented in terms of the Green’s function as

\[ w(x, t) = \int_0^R f(\xi)G(r, \xi, t)\,d\xi + \int_0^t \int_0^R \Phi(\xi, \tau)G(r, \xi, t-\tau)\,d\xi\,d\tau. \]

1.5-2. Domain: \( 0 \leq r \leq R \). First boundary value problem for the heat equation.

A boundary condition is prescribed:

\[ w = 0 \quad \text{at} \quad r = R. \]

Green’s function:

\[ G(r, \xi, t) = \sum_{n=1}^{\infty} \frac{2\xi}{R^2 J_1^2(\mu_n)} J_0\left( \frac{\mu_n r}{R} \right) J_0\left( \frac{\mu_n \xi}{R} \right) \exp\left( -\alpha \mu_n^2 t \frac{1}{R^2} \right), \]

where the \( \mu_n \) are positive zeros of the Bessel function, \( J_0(\mu) = 0 \). Below are the numerical values of the first ten roots:

\[
\begin{align*}
\mu_1 &= 2.4048, & \mu_2 &= 5.5201, & \mu_3 &= 8.6537, & \mu_4 &= 11.7915, & \mu_5 &= 14.9309, \\
\mu_6 &= 18.0711, & \mu_7 &= 21.2116, & \mu_8 &= 24.3525, & \mu_9 &= 27.4935, & \mu_{10} &= 30.6346.
\end{align*}
\]

The zeroes of the Bessel function \( J_0(\mu) \) may be approximated by the formula

\[ \mu_n = 2.4 + 3.13(n - 1) \quad (n = 1, 2, 3, \ldots), \]

which is accurate within 0.3%. As \( n \to \infty \), we have \( \mu_{n+1} - \mu_n \to \pi \).

1.5-3. Domain: \( 0 \leq r \leq R \). Second boundary value problem for the heat equation.

A boundary condition is prescribed:

\[ \frac{\partial w}{\partial r} = 0 \quad \text{at} \quad r = R. \]

Green’s function:

\[ G(r, \xi, t) = \frac{2}{R^2} \xi + \sum_{n=1}^{\infty} \frac{\xi}{R^2 J_0^2(\mu_n)} J_0\left( \frac{\mu_n r}{R} \right) J_0\left( \frac{\mu_n \xi}{R} \right) \exp\left( -\alpha \mu_n^2 t \frac{1}{R^2} \right), \]

where the \( \mu_n \) are positive zeros of the first-order Bessel function, \( J_1(\mu) = 0 \). Below are the numerical values of the first ten roots:

\[
\begin{align*}
\mu_1 &= 3.8317, & \mu_2 &= 7.0156, & \mu_3 &= 10.1735, & \mu_4 &= 13.3237, & \mu_5 &= 16.4706, \\
\mu_6 &= 19.6519, & \mu_7 &= 22.7601, & \mu_8 &= 25.9037, & \mu_9 &= 29.0468, & \mu_{10} &= 32.1897.
\end{align*}
\]

As \( n \to \infty \), we have \( \mu_{n+1} - \mu_n \to \pi \).
References


Nonhomogeneous Heat Equation with Axial Symmetry