1.7. Heat Equation of the Form $\dfrac{\partial w}{\partial t} = a\left(\dfrac{\partial^2 w}{\partial r^2} + \dfrac{2}{r}\dfrac{\partial w}{\partial r}\right) + \Phi(r,t)$

Nonhomogeneous heat (diffusion) equation with central symmetry.

1.7-1. Solutions of boundary value problems in terms of the Green's function.

We consider boundary value problems in domain $0 \le r \le R$ with the general initial condition

$$w = f(r) \quad \text{at} \quad t = 0$$

and various homogeneous boundary conditions (the solutions bounded at $r = 0$ are sought). The solution can be represented in terms of the Green's function as

$$w(x,t) = \int_0^R f(\xi) G(r,\xi,t)\,d\xi + \int_0^t \int_0^R \Phi(\xi,\tau) G(r,\xi,t-\tau)\,d\xi\,d\tau.$$

1.7-2. Domain: $0 \le r \le R$. First boundary value problem for heat equation.

A boundary condition is prescribed:

$$w = 0 \quad \text{at} \quad r = R.$$

Green's function:

$$G(r,\xi,t) = \frac{2\xi}{Rr} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi r}{R}\right) \sin\left(\frac{n\pi \xi}{R}\right) \exp\left(-\frac{an^2\pi^2 t}{R^2}\right).$$

1.7-3. Domain: $0 \le r \le R$. Second boundary value problem for heat equation.

A boundary condition is prescribed:

$$\frac{\partial w}{\partial r} = 0 \quad \text{at} \quad r = R.$$

Green's function:

$$G(r,\xi,t) = \frac{3\xi^2}{R^3} + \frac{2\xi}{Rr} \sum_{n=1}^{\infty} \frac{\mu_n^2 + 1}{\mu_n^2} \sin\left(\frac{\mu_n r}{R}\right) \sin\left(\frac{\mu_n \xi}{R}\right) \exp\left(-\frac{a\mu_n^2 t}{R^2}\right),$$

where the μ_n are positive roots of the transcendental equation $\tan\mu - \mu = 0$. The first five roots are

$$\mu_1 = 4.4934, \quad \mu_2 = 7.7253, \quad \mu_3 = 10.9041, \quad \mu_4 = 14.0662, \quad \mu_5 = 17.2208.$$

References

Carslaw, H. S. and Jaeger, J. C., *Conduction of Heat in Solids*, Clarendon Press, Oxford, 1984.

Polyanin, A. D., *Handbook of Linear Partial Differential Equations for Engineers and Scientists* , Chapman & Hall/CRC, 2002.

