2.1. Wave Equation $\dfrac{\partial^2 w}{\partial t^2} = a^2 \dfrac{\partial^2 w}{\partial x^2}$

This equation is also known as the ***equation of vibration of a string.*** The wave equation is often encountered in elasticity, aerodynamics, acoustics, and electrodynamics.

2.1-1. General solution. Some formulas.

1°. General solution:

$$w(x,t) = \varphi(x+at) + \psi(x-at),$$

where $\varphi(x)$ and $\psi(x)$ are arbitrary functions.

2°. If $w(x,t)$ is a solution of the wave equation, then the functions

$$\begin{aligned}
w_1 &= Aw(\pm\lambda x + C_1, \pm\lambda t + C_2) + B,\\
w_2 &= Aw\left(\frac{x - vt}{\sqrt{1-(v/a)^2}}, \frac{t - va^{-2}x}{\sqrt{1-(v/a)^2}}\right),\\
w_3 &= Aw\left(\frac{x}{x^2 - a^2t^2}, \frac{t}{x^2 - a^2t^2}\right),
\end{aligned}$$

are also solutions of the equation everywhere these functions are defined (A, B, C_1, C_2, v, and λ are arbitrary constants). The signs at λ's in the formula for w_1 are taken arbitrarily. The function w_2 results from the invariance of the wave equation under the Lorentz transformations.

2.1-2. Domain: $-\infty < x < \infty$. Cauchy problem for the wave equation.

Initial conditions are prescribed:

$$w = f(x) \quad \text{at} \quad t = 0, \qquad \frac{\partial w}{\partial t} = g(x) \quad \text{at} \quad t = 0.$$

Solution (D'Alembert's formula):

$$w(x,t) = \frac{1}{2}[f(x+at) + f(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} g(\xi)\,d\xi.$$

2.1-3. Domain: $0 \le x < \infty$. First boundary value problem for the wave equation.

The following two initial and one boundary conditions are prescribed:

$$w = f(x) \quad \text{at} \quad t = 0, \qquad \frac{\partial w}{\partial t} = g(x) \quad \text{at} \quad t = 0, \qquad w = h(t) \quad \text{at} \quad x = 0.$$

Solution:

$$w(x,t) = \begin{cases} \dfrac{1}{2}[f(x+at) + f(x-at)] + \dfrac{1}{2a}\displaystyle\int_{x-at}^{x+at} g(\xi)\,d\xi & \text{for } t < \dfrac{x}{a},\\[2ex] \dfrac{1}{2}[f(x+at) - f(at-x)] + \dfrac{1}{2a}\displaystyle\int_{at-x}^{x+at} g(\xi)\,d\xi + h\Big(t - \dfrac{x}{a}\Big) & \text{for } t > \dfrac{x}{a}. \end{cases}$$

In the domain $t < x/a$ the boundary conditions have no effect on the solution and the expression of $w(x,t)$ coincides with D'Alembert's solution for an infinite line (see Paragraph 2.1-2).

2.1-4. Domain: $0 \le x < \infty$. Second boundary value problem for the wave equation.

The following two initial and one boundary conditions are prescribed:

$$w = f(x) \quad \text{at} \quad t = 0, \qquad \frac{\partial w}{\partial t} = g(x) \quad \text{at} \quad t = 0, \qquad \frac{\partial w}{\partial x} = h(t) \quad \text{at} \quad x = 0.$$

Solution:

$$w(x,t) = \begin{cases} \frac{1}{2}[f(x+at) + f(x-at)] + \frac{1}{2a}[G(x+at) - G(x-at)] & \text{for } t < \frac{x}{a}, \\ \frac{1}{2}[f(x+at) + f(at-x)] + \frac{1}{2a}[G(x+at) + G(at-x)] - aH\Big(t - \frac{x}{a}\Big) & \text{for } t > \frac{x}{a}, \end{cases}$$

where $G(z) = \int_0^z g(\xi)\,d\xi$ and $H(z) = \int_0^z h(\xi)\,d\xi$.

2.1-5. Domain: $0 \le x \le l$. Boundary value problems for the wave equation.

For solutions of various boundary value problems, see the nonhomogeneous wave equation for $\Phi(x,t) \equiv 0$.

2.1-6. Other types of wave equations.

See also related linear equations:

- nonhomogeneous wave equation ,
- wave equation with axial symmetry ,
- wave equation with central symmetry ,
- Klein–Gordon equation ,
- nonhomogeneous Klein–Gordon equation ,
- telegraph equation .

References

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