2.1. Wave Equation \( \frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial^2 w}{\partial x^2} \)

This equation is also known as the equation of vibration of a string. The wave equation is often encountered in elasticity, aerodynamics, acoustics, and electrodynamics.


1°. General solution:
\[
w(x, t) = \varphi(x + at) + \psi(x - at),
\]
where \( \varphi(x) \) and \( \psi(x) \) are arbitrary functions.

2°. If \( w(x, t) \) is a solution of the wave equation, then the functions
\[
w_1 = A \varphi(\pm \lambda x + C_1 \pm \lambda t + C_2) + B,
\]
\[
w_2 = A \varphi \left( \frac{x - vt}{\sqrt{1 - (v/a)^2}}, \frac{t - v^2 x}{\sqrt{1 - (v/a)^2}} \right),
\]
\[
w_3 = A \varphi \left( \frac{x}{x^2 - a^2 t^2}, \frac{t}{x^2 - a^2 t^2} \right),
\]
are also solutions of the equation everywhere these functions are defined (\( A, B, C_1, C_2, v, \) and \( \lambda \) are arbitrary constants). The signs at \( \lambda \)'s in the formula for \( w_1 \) are taken arbitrarily. The function \( w_2 \) results from the invariance of the wave equation under the Lorentz transformations.

2.1-2. Domain: \(-\infty < x < \infty\). Cauchy problem for the wave equation.

Initial conditions are prescribed:
\[
w = f(x) \quad \text{at} \quad t = 0, \quad \frac{\partial w}{\partial t} = g(x) \quad \text{at} \quad t = 0.
\]

Solution (D’Alembert’s formula):
\[
w(x, t) = \frac{1}{2} [f(x + at) + f(x - at)] + \frac{1}{2a} \int_{x-at}^{x+at} g(\xi) \, d\xi.
\]

2.1-3. Domain: \( 0 \leq x < \infty \). First boundary value problem for the wave equation.

The following two initial and one boundary conditions are prescribed:
\[
w = f(x) \quad \text{at} \quad t = 0, \quad \frac{\partial w}{\partial t} = g(x) \quad \text{at} \quad t = 0, \quad w = h(t) \quad \text{at} \quad x = 0.
\]

Solution:
\[
w(x, t) = \begin{cases}
\frac{1}{2} [f(x + at) + f(x - at)] + \frac{1}{2a} \int_{x-at}^{x+at} g(\xi) \, d\xi & \text{for } t < \frac{x}{a}, \\
\frac{1}{2} [f(x + at) - f(at - x)] + \frac{1}{2a} \int_{at-x}^{at+x} g(\xi) \, d\xi + h\left( t - \frac{x}{a} \right) & \text{for } t > \frac{x}{a}.
\end{cases}
\]

In the domain \( t < x/a \) the boundary conditions have no effect on the solution and the expression of \( w(x, t) \) coincides with D’Alembert’s solution for an infinite line (see Paragraph 2.1-2).
2.1-4. Domain: \( 0 \leq x < \infty \). Second boundary value problem for the wave equation.

The following two initial and one boundary conditions are prescribed:

\[
    w = f(x) \quad \text{at} \quad t = 0, \quad \frac{\partial w}{\partial t} = g(x) \quad \text{at} \quad t = 0, \quad \frac{\partial w}{\partial x} = h(t) \quad \text{at} \quad x = 0.
\]

Solution:

\[
    w(x, t) = \begin{cases} 
        \frac{1}{2} \left[ f(x + at) + f(x - at) \right] + \frac{1}{2a} \left[ G(x + at) - G(x - at) \right] & \text{for} \quad t < \frac{x}{a}, \\
        \frac{1}{2} \left[ f(x + at) + f(at - x) \right] + \frac{1}{2a} \left[ G(x + at) + G(at - x) \right] - aH \left( t - \frac{x}{a} \right) & \text{for} \quad t > \frac{x}{a},
    \end{cases}
\]

where \( G(z) = \int_0^z g(\xi) \, d\xi \) and \( H(z) = \int_0^z h(\xi) \, d\xi \).

2.1-5. Domain: \( 0 \leq x \leq l \). Boundary value problems for the wave equation.

For solutions of various boundary value problems, see the [nonhomogeneous wave equation] for \( \Phi(x, t) = 0 \).

2.1-6. Other types of wave equations.

See also related linear equations:

- [nonhomogeneous wave equation],
- wave equation with axial symmetry,
- wave equation with central symmetry,
- Klein–Gordon equation,
- nonhomogeneous Klein–Gordon equation,
- telegraph equation.

References


