2.2. Nonhomogeneous Wave Equation \( \frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial^2 w}{\partial x^2} + \Phi(x, t) \)

2.2-1. Solutions of boundary value problems in terms of the Green’s function.

We consider boundary value problems for the nonhomogeneous wave equation on a finite interval \( 0 \leq x \leq l \) with the general initial conditions

\[ w = f(x) \quad \text{at} \quad t = 0, \quad \frac{\partial w}{\partial t} = g(x) \quad \text{at} \quad t = 0 \]

and various homogeneous boundary conditions. The solution can be represented in terms of the Green’s function as

\[ w(x, t) = \int_0^l f(\xi)G(x, \xi, t)\,d\xi + \int_0^l g(\xi)G(x, \xi, t)\,d\xi + \int_0^t \int_0^l \Phi(\xi, \tau)G(x, \xi, t-\tau)\,d\xi\,d\tau. \]

2.2-2. Domain: \( 0 \leq x \leq l \). First boundary value problem for the wave equation.

Boundary conditions are prescribed:

\[ w = 0 \quad \text{at} \quad x = 0, \quad w = 0 \quad \text{at} \quad x = l. \]

Green’s function:

\[ G(x, \xi, t) = \frac{2}{a\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \left( \frac{n\pi x}{l} \right) \sin \left( \frac{n\pi \xi}{l} \right) \sin \left( \frac{n\pi at}{l} \right). \]

2.2-3. Domain: \( 0 \leq x \leq l \). Second boundary value problem for the wave equation.

Boundary conditions are prescribed:

\[ \frac{\partial w}{\partial x} = 0 \quad \text{at} \quad x = 0, \quad \frac{\partial w}{\partial x} = 0 \quad \text{at} \quad x = l. \]

Green’s function:

\[ G(x, \xi, t) = \frac{t}{l} + \frac{2}{a\pi} \sum_{n=1}^{\infty} \frac{1}{n} \cos \left( \frac{n\pi x}{l} \right) \cos \left( \frac{n\pi \xi}{l} \right) \sin \left( \frac{n\pi at}{l} \right). \]

2.2-4. Domain: \( 0 \leq x \leq l \). Third boundary value problem \( (k_1 > 0, k_2 > 0) \).

Boundary conditions are prescribed:

\[ \frac{\partial w}{\partial x} - k_1 w = 0 \quad \text{at} \quad x = 0, \quad \frac{\partial w}{\partial x} + k_2 w = 0 \quad \text{at} \quad x = l. \]

Green’s function:

\[ G(x, \xi, t) = \frac{1}{a} \sum_{n=1}^{\infty} \frac{1}{\lambda_n ||u_n||^2} \sin(\lambda_n x + \varphi_n) \sin(\lambda_n \xi + \varphi_n) \sin(\lambda_n at), \]

\[ \varphi_n = \arctan \frac{\lambda_n}{k_1}, \quad ||u_n||^2 = \frac{1}{2} + \frac{(\lambda_n^2 + k_1 k_2)(k_1 + k_2)}{2(\lambda_n^2 + k_1^2)(\lambda_n^2 + k_2^2)}, \]

the \( \lambda_n \) are positive roots of the transcendental equation \( \cot(\lambda l) = \frac{\lambda^2 - k_1 k_2}{\lambda(k_1 + k_2)}. \)
References


Linear Nonhomogeneous Wave Equation

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