2.3. Klein–Gordon Equation $\dfrac{\partial^2 w}{\partial t^2} = a^2 \dfrac{\partial^2 w}{\partial x^2} - bw$

The Klein–Gordon equation is encountered in quantum field theory and a number of applications.

2.3-1. Particular solutions of the Klein–Gordon equation:

$$w(x,t) = \cos(\lambda x)[A\cos(\mu t) + B\sin(\mu t)], \quad b = -a^2\lambda^2 + \mu^2,$$
$$w(x,t) = \sin(\lambda x)[A\cos(\mu t) + B\sin(\mu t)], \quad b = -a^2\lambda^2 + \mu^2,$$
$$w(x,t) = \exp(\pm\mu t)[A\cos(\lambda x) + B\sin(\lambda x)], \quad b = -a^2\lambda^2 - \mu^2,$$
$$w(x,t) = \exp(\pm\lambda x)[A\cos(\mu t) + B\sin(\mu t)], \quad b = a^2\lambda^2 + \mu^2,$$
$$w(x,t) = \exp(\pm\lambda x)[A\exp(\mu t) + B\exp(-\mu t)], \quad b = a^2\lambda^2 - \mu^2,$$
$$w(x,t) = AJ_0(\xi) + BY_0(\xi), \quad \xi = \frac{\sqrt{b}}{a}\sqrt{a^2(t+C_1)^2 - (x+C_2)^2}, \quad b > 0,$$
$$w(x,t) = AI_0(\xi) + BK_0(\xi), \quad \xi = \frac{\sqrt{-b}}{a}\sqrt{a^2(t+C_1)^2 - (x+C_2)^2}, \quad b < 0,$$

where A, B, C_1, and C_2 are arbitrary constants, $J_0(\xi)$ and $Y_0(\xi)$ are the Bessel functions, and $I_0(\xi)$ and $K_0(\xi)$ are the modified Bessel functions.

2.3-2. Formulas allowing the construction of particular solutions.

Suppose $w = w(x,t)$ is a solution of the Klein–Gordon equation. Then the functions

$$w_1 = Aw(\pm x + C_1, \pm t + C_2) + B,$$
$$w_2 = Aw\left(\frac{x - vt}{\sqrt{1-(v/a)^2}}, \frac{t - va^{-2}x}{\sqrt{1-(v/a)^2}}\right),$$

where A, B, C_1, C_2, and v are arbitrary constants, are also solutions of this equation. The signs in the formula for w_1 are taken arbitrarily.

2.3-3. Domain: $0 \le x \le l$. Boundary value problems for the Klein–Gordon equation.

For solutions of the first and second boundary value problems, see the nonhomogeneous Klein–Gordon equation for $\Phi(x,t) \equiv 0$.

References

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