2.3. **Klein–Gordon Equation** \( \frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial^2 w}{\partial x^2} - bw \)

The Klein–Gordon equation is encountered in quantum field theory and a number of applications.

2.3-1. **Particular solutions of the Klein–Gordon equation:**

\[
\begin{align*}
  w(x, t) &= \cos(\lambda x)[A \cos(\mu t) + B \sin(\mu t)], \quad b = -a^2 \lambda^2 + \mu^2, \\
  w(x, t) &= \sin(\lambda x)[A \cos(\mu t) + B \sin(\mu t)], \quad b = -a^2 \lambda^2 + \mu^2, \\
  w(x, t) &= \exp(\pm \mu t)[A \cos(\lambda x) + B \sin(\lambda x)], \quad b = -a^2 \lambda^2 - \mu^2, \\
  w(x, t) &= \exp(\pm \lambda x)[A \cos(\mu t) + B \sin(\mu t)], \quad b = a^2 \lambda^2 + \mu^2, \\
  w(x, t) &= \exp(\pm \lambda x)[A \exp(\mu t) + B \exp(-\mu t)], \quad b = a^2 \lambda^2 - \mu^2, \\
  w(x, t) &= AJ_0(\xi) + BY_0(\xi), \quad \xi = \frac{\sqrt{b}}{a} \sqrt{a^2 (t + C_1)^2 - (x + C_2)^2}, \quad b > 0, \\
  w(x, t) &= AI_0(\xi) + BK_0(\xi), \quad \xi = -\frac{\sqrt{-b}}{a} \sqrt{a^2 (t + C_1)^2 - (x + C_2)^2}, \quad b < 0,
\end{align*}
\]

where \( A, B, C_1, \) and \( C_2 \) are arbitrary constants, \( J_0(\xi) \) and \( Y_0(\xi) \) are the Bessel functions, and \( I_0(\xi) \) and \( K_0(\xi) \) are the modified Bessel functions.

2.3-2. **Formulas allowing the construction of particular solutions.**

Suppose \( w = w(x, t) \) is a solution of the Klein–Gordon equation. Then the functions

\[
\begin{align*}
  w_1 &= Aw(\pm x + C_1, \pm t + C_2) + B, \\
  w_2 &= Aw\left(\frac{x - vt}{\sqrt{1 - (v/a)^2}}, \frac{t - vq^2 x}{\sqrt{1 - (v/a)^2}}\right),
\end{align*}
\]

where \( A, B, C_1, C_2, \) and \( v \) are arbitrary constants, are also solutions of this equation. The signs in the formula for \( w_1 \) are taken arbitrarily.

2.3-3. **Domain: \( 0 \leq x \leq l \). Boundary value problems for the Klein–Gordon equation.**

For solutions of the first and second boundary value problems, see the [nonhomogeneous Klein–Gordon equation] for \( \Phi(x, t) \equiv 0 \).

**References**
