2.4. Nonhomogeneous Klein–Gordon Equation

$$\frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial^2 w}{\partial x^2} - bw + \Phi(x,t)$$

2.4-1. Solutions of boundary value problems in terms of the Green's function.

We consider boundary value problems for the nonhomogeneous Klein–Gordon equation on a finite interval $0 \le x \le l$ with the general initial conditions

$$w = f(x) \quad \text{at} \quad t = 0, \qquad \frac{\partial w}{\partial t} = g(x) \quad \text{at} \quad t = 0$$

and various homogeneous boundary conditions. The solution can be represented in terms of the Green's function as

$$w(x,t) = \frac{\partial}{\partial t} \int_0^l f(\xi) G(x,\xi,t)\, d\xi + \int_0^l g(\xi) G(x,\xi,t)\, d\xi + \int_0^t \int_0^l \Phi(\xi,\tau) G(x,\xi,t-\tau)\, d\xi\, d\tau.$$

2.4-2. Domain: $0 \le x \le l$. First boundary value problem for the Klein–Gordon equation.

Boundary conditions are prescribed:

$$w = 0 \quad \text{at} \quad x = 0, \qquad w = 0 \quad \text{at} \quad x = l.$$

Green's function for $b > 0$:

$$G(x,\xi,t) = \frac{2}{l} \sum_{n=1}^{\infty} \sin(\lambda_n x) \sin(\lambda_n \xi) \frac{\sin\bigl(t\sqrt{a^2\lambda_n^2 + b}\bigr)}{\sqrt{a^2\lambda_n^2 + b}}, \qquad \lambda_n = \frac{\pi n}{l}.$$

2.4-3. Domain: $0 \le x \le l$. Second boundary value problem for the Klein–Gordon equation.

Boundary conditions are prescribed:

$$\frac{\partial w}{\partial x} = 0 \quad \text{at} \quad x = 0, \qquad \frac{\partial w}{\partial x} = 0 \quad \text{at} \quad x = l.$$

Green's function for $b > 0$:

$$G(x,\xi,t) = \frac{1}{l\sqrt{b}} \sin\bigl(t\sqrt{b}\bigr) + \frac{2}{l} \sum_{n=1}^{\infty} \cos(\lambda_n x) \cos(\lambda_n \xi) \frac{\sin\bigl(t\sqrt{a^2\lambda_n^2 + b}\bigr)}{\sqrt{a^2\lambda_n^2 + b}}, \qquad \lambda_n = \frac{\pi n}{l}.$$

References

Butkovskiy, A. G., *Green's Functions and Transfer Functions Handbook*, Halstead Press–John Wiley & Sons, New York, 1982.

Polyanin, A. D., *Handbook of Linear Partial Differential Equations for Engineers and Scientists*, Chapman & Hall/CRC, 2002.

