2.4. Nonhomogeneous Klein–Gordon Equation

\[
\frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial^2 w}{\partial x^2} - bw + \Phi(x, t)
\]

2.4-1. Solutions of boundary value problems in terms of the Green’s function.

We consider boundary value problems for the nonhomogeneous Klein–Gordon equation on a finite interval \(0 \leq x \leq l\) with the general initial conditions

\[
w = f(x) \quad \text{at} \quad t = 0, \quad \frac{\partial w}{\partial t} = g(x) \quad \text{at} \quad t = 0
\]

and various homogeneous boundary conditions. The solution can be represented in terms of the Green’s function as

\[
w(x, t) = \frac{\partial}{\partial t} \int_0^l f(\xi) G(x, \xi, t) \, d\xi + \int_0^l g(\xi) G(x, \xi, t) \, d\xi + \int_0^t \int_0^l \Phi(\xi, \tau) G(x, \xi, t - \tau) \, d\xi \, d\tau.
\]

2.4-2. Domain: \(0 \leq x \leq l\). First boundary value problem for the Klein–Gordon equation.

Boundary conditions are prescribed:

\[
w = 0 \quad \text{at} \quad x = 0, \quad w = 0 \quad \text{at} \quad x = l.
\]

Green’s function for \(b > 0\):

\[
G(x, \xi, t) = 2 \frac{\sum_{n=1}^{\infty} \sin(\lambda_n x) \sin(\lambda_n \xi)}{l} \frac{\sin\left(t\sqrt{a^2\lambda_n^2 + b}\right)}{\sqrt{a^2\lambda_n^2 + b}}, \quad \lambda_n = \frac{\pi n}{l}.
\]

2.4-3. Domain: \(0 \leq x \leq l\). Second boundary value problem for the Klein–Gordon equation.

Boundary conditions are prescribed:

\[
\frac{\partial w}{\partial x} = 0 \quad \text{at} \quad x = 0, \quad \frac{\partial w}{\partial x} = 0 \quad \text{at} \quad x = l.
\]

Green’s function for \(b > 0\):

\[
G(x, \xi, t) = \frac{1}{l \sqrt{b}} \sin\left(t\sqrt{b}\right) + 2 \frac{\sum_{n=1}^{\infty} \cos(\lambda_n x) \cos(\lambda_n \xi) \sin\left(t\sqrt{a^2\lambda_n^2 + b}\right)}{\sqrt{a^2\lambda_n^2 + b}}, \quad \lambda_n = \frac{\pi n}{l}.
\]

References
