2.5. Wave Equation of the Form $\frac{\partial^2 w}{\partial t^2} = a^2 \left( \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right) + \Phi(r, t)$

This is the wave equation with axial symmetry, where $r = \sqrt{x^2 + y^2}$ is the radial coordinate.

2.5-1. Solutions of boundary value problems in terms of the Green’s function.

We consider boundary value problems for the wave equation with axial symmetry in domain $0 \leq r \leq R$ with the general initial conditions

$$w = f(r) \quad \text{at} \quad t = 0, \quad \frac{\partial w}{\partial t} = g(r) \quad \text{at} \quad t = 0$$

and various homogeneous boundary conditions at $r = R$ (the solutions bounded at $r = 0$ are sought). The solution can be represented in terms of the Green’s function as

$$w(r, t) = \frac{\partial}{\partial t} \int_0^R f(\xi)G(r, \xi, t)\,d\xi + \int_0^R g(\xi)G(r, \xi, t)\,d\xi + \int_0^1 \int_0^R \Phi(\xi, \tau)G(r, \xi, t - \tau)\,d\xi\,d\tau.$$

2.5-2. Domain: $0 \leq r \leq R$. First boundary value problem for the wave equation.

A boundary condition is prescribed:

$$w = 0 \quad \text{at} \quad r = R.$$

Green’s function:

$$G(r, \xi, t) = \frac{2\xi}{aR} \sum_{n=1}^\infty \frac{1}{\lambda_n J_1^2(\lambda_n)} J_0 \left( \frac{\lambda_n r}{R} \right) J_0 \left( \frac{\lambda_n \xi}{R} \right) \sin \left( \frac{\lambda_n \omega t}{R} \right),$$

where the $\lambda_n$ are positive zeros of the Bessel function, $J_0(\lambda) = 0$. Below are the numerical values of the first ten roots:

$$\lambda_1 = 2.4048, \quad \lambda_2 = 5.5201, \quad \lambda_3 = 8.6537, \quad \lambda_4 = 11.7915, \quad \lambda_5 = 14.9309,$$

$$\lambda_6 = 18.0711, \quad \lambda_7 = 21.2116, \quad \lambda_8 = 24.3525, \quad \lambda_9 = 27.4935, \quad \lambda_{10} = 30.6346.$$

2.5-3. Domain: $0 \leq r \leq R$. Second boundary value problem for the wave equation

A boundary condition is prescribed:

$$\frac{\partial w}{\partial r} = 0 \quad \text{at} \quad r = R.$$

Green’s function:

$$G(r, \xi, t) = \frac{2\xi}{aR^2} + \frac{2\xi}{\lambda_n} \sum_{n=1}^\infty \frac{1}{\lambda_n J_0^2(\lambda_n)} J_0 \left( \frac{\lambda_n r}{R} \right) J_0 \left( \frac{\lambda_n \xi}{R} \right) \sin \left( \frac{\lambda_n \omega t}{R} \right),$$

where the $\lambda_n$ are positive zeros of the first-order Bessel function, $J_1(\lambda) = 0$. Below are the numerical values of the first ten roots:

$$\lambda_1 = 3.8317, \quad \lambda_2 = 7.0156, \quad \lambda_3 = 10.1735, \quad \lambda_4 = 13.3237, \quad \lambda_5 = 16.4706,$$

$$\lambda_6 = 19.6159, \quad \lambda_7 = 22.7601, \quad \lambda_8 = 25.9037, \quad \lambda_9 = 29.0468, \quad \lambda_{10} = 32.1897.$$

References


Wave Equation with Axial Symmetry

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