2.5. Wave Equation of the Form $\dfrac{\partial^2 w}{\partial t^2} = a^2\left(\dfrac{\partial^2 w}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial w}{\partial r}\right) + \Phi(r,t)$

This is the ***wave equation with axial symmetry***, where $r = \sqrt{x^2+y^2}$ is the radial coordinate.

2.5-1. Solutions of boundary value problems in terms of the Green's function.

We consider boundary value problems for the wave equation with axial symmetry in domain $0 \le r \le R$ with the general initial conditions

$$w = f(r) \quad \text{at} \quad t = 0, \qquad \frac{\partial w}{\partial t} = g(r) \quad \text{at} \quad t = 0$$

and various homogeneous boundary conditions at $r = R$ (the solutions bounded at $r = 0$ are sought). The solution can be represented in terms of the Green's function as

$$w(r,t) = \frac{\partial}{\partial t}\int_0^R f(\xi)G(r,\xi,t)\,d\xi + \int_0^R g(\xi)G(r,\xi,t)\,d\xi + \int_0^t\int_0^R \Phi(\xi,\tau)G(r,\xi,t-\tau)\,d\xi\,d\tau.$$

2.5-2. Domain: $0 \le r \le R$. First boundary value problem for the wave equation.

A boundary condition is prescribed:

$$w = 0 \quad \text{at} \quad r = R.$$

Green's function:

$$G(r,\xi,t) = \frac{2\xi}{aR}\sum_{n=1}^{\infty}\frac{1}{\lambda_n J_1^2(\lambda_n)} J_0\left(\frac{\lambda_n r}{R}\right) J_0\left(\frac{\lambda_n \xi}{R}\right)\sin\left(\frac{\lambda_n a t}{R}\right),$$

where the λ_n are positive zeros of the Bessel function, $J_0(\lambda) = 0$. Below are the numerical values of the first ten roots:

$$\mu_1 = 2.4048, \quad \mu_2 = 5.5201, \quad \mu_3 = 8.6537, \quad \mu_4 = 11.7915, \quad \mu_5 = 14.9309,$$
$$\mu_6 = 18.0711, \quad \mu_7 = 21.2116, \quad \mu_8 = 24.3525, \quad \mu_9 = 27.4935, \quad \mu_{10} = 30.6346.$$

2.5-3. Domain: $0 \le r \le R$. Second boundary value problem for the wave equation

A boundary condition is prescribed:

$$\frac{\partial w}{\partial r} = 0 \quad \text{at} \quad r = R.$$

Green's function:

$$G(r,\xi,t) = \frac{2t\xi}{R^2} + \frac{2\xi}{aR}\sum_{n=1}^{\infty}\frac{1}{\lambda_n J_0^2(\lambda_n)} J_0\left(\frac{\lambda_n r}{R}\right) J_0\left(\frac{\lambda_n \xi}{R}\right)\sin\left(\frac{\lambda_n a t}{R}\right),$$

where the λ_n are positive zeros of the first-order Bessel function, $J_1(\lambda) = 0$. Below are the numerical values of the first ten roots:

$$\mu_1 = 3.8317, \quad \mu_2 = 7.0156, \quad \mu_3 = 10.1735, \quad \mu_4 = 13.3237, \quad \mu_5 = 16.4706,$$
$$\mu_6 = 19.6159, \quad \mu_7 = 22.7601, \quad \mu_8 = 25.9037, \quad \mu_9 = 29.0468, \quad \mu_{10} = 32.1897.$$

References

Budak, B. M., Samarskii, A. A., and Tikhonov, A. N., *Collection of Problems on Mathematical Physics* [in Russian], Nauka, Moscow, 1980.

Polyanin, A. D., *Handbook of Linear Partial Differential Equations for Engineers and Scientists* , Chapman & Hall/CRC, 2002.

