2.6. Wave Equation of Form \[ \frac{\partial^2 w}{\partial t^2} = a^2 \left( \frac{\partial^2 w}{\partial r^2} + \frac{2}{r} \frac{\partial w}{\partial r} \right) + \Phi(r, t) \]

This is the wave equation with central symmetry or the equation of vibration of a gas with central symmetry, where \( r = \sqrt{x^2 + y^2 + z^2} \) is the radial coordinate.

2.6-1. General solution for \( \Phi(r, t) \equiv 0 \):

\[ w(t, r) = \frac{\varphi(r + at) + \psi(r - at)}{r}, \]

where \( \varphi(r_1) \) and \( \psi(r_2) \) are arbitrary functions.

2.6-2. Reduction to a constant coefficient wave equation.

The substitution \( u(r, t) = rw(r, t) \) leads to the nonhomogeneous constant coefficient wave equation

\[ \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial r^2} + r\Phi(r, t), \]

which is discussed in [Subsection 2.1] (with \( \Phi \equiv 0 \)) and in [Subsection 2.2] (with \( \Phi \neq 0 \)).

2.6-3. Solutions of boundary value problems in terms of the Green’s function.

We consider boundary value problems for the wave equation with central symmetry in domain \( 0 \leq r \leq R \) with the general initial conditions

\[ w = f(r) \quad \text{at} \quad t = 0, \quad \frac{\partial w}{\partial t} = g(r) \quad \text{at} \quad t = 0 \]

and various homogeneous boundary conditions at \( r = R \) (the solutions bounded at \( r = 0 \) are sought). The solution can be represented in terms of the Green’s function as

\[ w(r, t) = \frac{\partial}{\partial t} \int_0^R f(\xi)G(r, \xi, t) \, d\xi + \int_0^R g(\xi)G(r, \xi, t) \, d\xi + \int_0^t \int_0^R \Phi(\xi, \tau)G(r, \xi, t - \tau) \, d\xi \, d\tau. \]

2.6-4. Domain: \( 0 \leq r \leq R \). First boundary value problem for the wave equation.

A boundary condition is prescribed:

\[ w = 0 \quad \text{at} \quad r = R. \]

Green’s function:

\[ G(r, \xi, t) = \frac{2\xi}{\pi ar} \sum_{n=1}^{\infty} \frac{1}{n} \sin \left( \frac{n\pi r}{R} \right) \sin \left( \frac{n\pi \xi}{R} \right) \sin \left( \frac{an\pi t}{R} \right). \]

2.6-5. Domain: \( 0 \leq r \leq R \). Second boundary value problem for the wave equation.

A boundary condition is prescribed:

\[ \frac{\partial w}{\partial r} = 0 \quad \text{at} \quad r = R. \]

Green’s function:

\[ G(r, \xi, t) = \frac{3\xi^2}{R^3} + \frac{2\xi}{ar} \sum_{n=1}^{\infty} \frac{\mu_n^2 + 1}{\mu_n^3} \sin \left( \frac{\mu_n r}{R} \right) \sin \left( \frac{\mu_n \xi}{R} \right) \sin \left( \frac{\mu_n at}{R} \right), \]

where the \( \mu_n \) are positive roots of the transcendental equation \( \tan \mu - \mu = 0 \).
References
