2.6. Wave Equation of Form $\dfrac{\partial^2 w}{\partial t^2} = a^2\left(\dfrac{\partial^2 w}{\partial r^2} + \dfrac{2}{r}\dfrac{\partial w}{\partial r}\right) + \Phi(r,t)$

This is the ***wave equation with central symmetry*** or the ***equation of vibration of a gas with central symmetry***, where $r = \sqrt{x^2+y^2+z^2}$ is the radial coordinate.

2.6-1. General solution for $\Phi(r,t) \equiv 0$:

$$w(t,r) = \frac{\varphi(r+at) + \psi(r-at)}{r},$$

where $\varphi(r_1)$ and $\psi(r_2)$ are arbitrary functions.

2.6-2. Reduction to a constant coefficient wave equation.

The substitution $u(r,t) = rw(r,t)$ leads to the nonhomogeneous constant coefficient wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial r^2} + r\Phi(r,t),$$

which is discussed in Subsection 2.1 (with $\Phi \equiv 0$) and in Subsection 2.2 (with $\Phi \not\equiv 0$).

2.6-3. Solutions of boundary value problems in terms of the Green's function.

We consider boundary value problems for the wave equation with central symmetry in domain $0 \le r \le R$ with the general initial conditions

$$w = f(r) \quad \text{at} \quad t = 0, \qquad \frac{\partial w}{\partial t} = g(r) \quad \text{at} \quad t = 0$$

and various homogeneous boundary conditions at $r = R$ (the solutions bounded at $r = 0$ are sought). The solution can be represented in terms of the Green's function as

$$w(r,t) = \frac{\partial}{\partial t}\int_0^R f(\xi)G(r,\xi,t)\,d\xi + \int_0^R g(\xi)G(r,\xi,t)\,d\xi + \int_0^t\int_0^R \Phi(\xi,\tau)G(r,\xi,t-\tau)\,d\xi\,d\tau.$$

2.6-4. Domain: $0 \le r \le R$. First boundary value problem for the wave equation.

A boundary condition is prescribed:

$$w = 0 \quad \text{at} \quad r = R.$$

Green's function:

$$G(r,\xi,t) = \frac{2\xi}{\pi a r}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi r}{R}\right)\sin\left(\frac{n\pi\xi}{R}\right)\sin\left(\frac{an\pi t}{R}\right).$$

2.6-5. Domain: $0 \le r \le R$. Second boundary value problem for the wave equation.

A boundary condition is prescribed:

$$\frac{\partial w}{\partial r} = 0 \quad \text{at} \quad r = R.$$

Green's function:

$$G(r,\xi,t) = \frac{3t\xi^2}{R^3} + \frac{2\xi}{ar}\sum_{n=1}^{\infty}\frac{\mu_n^2+1}{\mu_n^3}\sin\left(\frac{\mu_n r}{R}\right)\sin\left(\frac{\mu_n\xi}{R}\right)\sin\left(\frac{\mu_n a t}{R}\right),$$

where the μ_n are positive roots of the transcendental equation $\tan\mu - \mu = 0$.

References

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