5.2. Equation of the Form $\dfrac{\partial^2 w}{\partial t^2} + a^2 \dfrac{\partial^4 w}{\partial x^4} = \Phi(x,t)$

Nonhomogeneous equation of transverse vibration of elastic rods.

5.2-1. Domain: $0 \le x \le l$. Solution in terms of the Green's function.

We consider boundary value problems on an interval $0 \le x \le l$ with the general initial condition

$$w = f(x) \quad \text{at} \quad t = 0, \qquad \partial_t w = g(x) \quad \text{at} \quad t = 0$$

and various homogeneous boundary conditions. The solution can be represented in terms of the Green's function as

$$w(x,t) = \frac{\partial}{\partial t}\int_0^l f(\xi)G(x,\xi,t)\,d\xi + \int_0^l g(\xi)G(x,\xi,t)\,d\xi + \int_0^t\int_0^l \Phi(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau.$$

5.2-2. Both ends of the rod are clamped.

Boundary conditions are prescribed:

$$w = \partial_x w = 0 \quad \text{at} \quad x = 0, \qquad w = \partial_x w = 0 \quad \text{at} \quad x = l.$$

Green's function:

$$G(x,\xi,t) = \frac{4}{al}\sum_{n=1}^{\infty}\frac{\lambda_n^2}{\bigl[\varphi_n''(l)\bigr]^2}\varphi_n(x)\varphi_n(\xi)\sin(\lambda_n^2 at),$$

where

$$\varphi_n(x) = \bigl[\sinh(\lambda_n l) - \sin(\lambda_n l)\bigr]\bigl[\cosh(\lambda_n x) - \cos(\lambda_n x)\bigr] - \bigl[\cosh(\lambda_n l) - \cos(\lambda_n l)\bigr]\bigl[\sinh(\lambda_n x) - \sin(\lambda_n x)\bigr];$$

the λ_n are positive roots of the transcendental equation $\cosh(\lambda l)\cos(\lambda l) = 1$. The numerical values of the roots can be calculated from the formulas

$$\lambda_n = \frac{\mu_n}{l}, \quad \text{where} \quad \mu_1 = 1.875, \quad \mu_2 = 4.694, \quad \mu_n = \frac{\pi}{2}(2n-1) \quad \text{for} \quad n \ge 3.$$

5.2-3. Both ends of the rod are hinged.

Boundary conditions are prescribed:

$$w = \partial_{xx} w = 0 \quad \text{at} \quad x = 0, \qquad w = \partial_{xx} w = 0 \quad \text{at} \quad x = l.$$

Green's function:

$$G(x,\xi,t) = \frac{2l}{a\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin(\lambda_n x)\sin(\lambda_n \xi)\sin(\lambda_n^2 at), \qquad \lambda_n = \frac{\pi n}{l}.$$

5.2-4. One end of the rod is clamped and the other is hinged.

Boundary conditions are prescribed:

$$w = \partial_x w = 0 \quad \text{at} \quad x = 0, \qquad w = \partial_{xx} w = 0 \quad \text{at} \quad x = l.$$

Green's function:

$$G(x,\xi,t)=\frac{2}{al}\sum_{n=1}^{\infty}\lambda_n^2\frac{\varphi_n(x)\varphi_n(\xi)}{|\varphi_n'(l)\varphi_n'''(l)|}\sin(\lambda_n^2 at),$$

where

$$\varphi_n(x)=\bigl[\sinh(\lambda_n l)-\sin(\lambda_n l)\bigr]\bigl[\cosh(\lambda_n x)-\cos(\lambda_n x)\bigr]-\bigl[\cosh(\lambda_n l)-\cos(\lambda_n l)\bigr]\bigl[\sinh(\lambda_n x)-\sin(\lambda_n x)\bigr];$$

the λ_n are positive roots of the transcendental equation $\tan(\lambda l)-\tanh(\lambda l)=0$.

5.2-5. One end of the rod is clamped and the other is free.

Boundary conditions are prescribed:

$$w=\partial_x w=0 \quad \text{at} \quad x=0, \qquad \partial_{xx}w=\partial_{xxx}w=0 \quad \text{at} \quad x=l.$$

Green's function:

$$G(x,\xi,t)=\frac{4}{al}\sum_{n=1}^{\infty}\frac{\varphi_n(x)\varphi_n(\xi)}{\lambda_n^2\varphi_n^2(l)}\sin(\lambda_n^2 at),$$

where

$$\varphi_n(x)=\bigl[\sinh(\lambda_n l)+\sin(\lambda_n l)\bigr]\bigl[\cosh(\lambda_n x)-\cos(\lambda_n x)\bigr]-\bigl[\cosh(\lambda_n l)+\cos(\lambda_n l)\bigr]\bigl[\sinh(\lambda_n x)-\sin(\lambda_n x)\bigr];$$

the λ_n are positive roots of the transcendental equation $\cosh(\lambda l)\cos(\lambda l)=-1$.

5.2-6. One end of the rod is hinged and the other is free.

Boundary conditions are prescribed:

$$w=\partial_{xx} w=0 \quad \text{at} \quad x=0, \qquad \partial_{xx}w=\partial_{xxx}w=0 \quad \text{at} \quad x=l.$$

Green's function:

$$G(x,\xi,t)=\frac{4}{al}\sum_{n=1}^{\infty}\frac{\varphi_n(x)\varphi_n(\xi)}{\lambda_n^2\varphi_n^2(l)}\sin(\lambda_n^2 at),$$

where

$$\varphi_n(x)=\sin(\lambda_n l)\sinh(\lambda_n x)+\sinh(\lambda_n l)\sin(\lambda_n x);$$

the λ_n are positive roots of the transcendental equation $\tan(\lambda l)-\tanh(\lambda l)=0$.

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