



## 5.2. Equation of the Form $\frac{\partial^2 w}{\partial t^2} + a^2 \frac{\partial^4 w}{\partial x^4} = \Phi(x, t)$

*Nonhomogeneous equation of transverse vibration of elastic rods.*

### 5.2-1. Domain: $0 \leq x \leq l$ . Solution in terms of the Green's function.

We consider boundary value problems on an interval  $0 \leq x \leq l$  with the general initial condition

$$w = f(x) \quad \text{at} \quad t = 0, \quad \partial_t w = g(x) \quad \text{at} \quad t = 0$$

and various homogeneous boundary conditions. The solution can be represented in terms of the Green's function as

$$w(x, t) = \frac{\partial}{\partial t} \int_0^l f(\xi) G(x, \xi, t) d\xi + \int_0^l g(\xi) G(x, \xi, t) d\xi + \int_0^t \int_0^l \Phi(\xi, \tau) G(x, \xi, t - \tau) d\xi d\tau.$$

### 5.2-2. Both ends of the rod are clamped.

Boundary conditions are prescribed:

$$w = \partial_x w = 0 \quad \text{at} \quad x = 0, \quad w = \partial_x w = 0 \quad \text{at} \quad x = l.$$

Green's function:

$$G(x, \xi, t) = \frac{4}{al} \sum_{n=1}^{\infty} \frac{\lambda_n^2}{[\varphi_n''(l)]^2} \varphi_n(x) \varphi_n(\xi) \sin(\lambda_n^2 at),$$

where

$$\varphi_n(x) = [\sinh(\lambda_n l) - \sin(\lambda_n l)] [\cosh(\lambda_n x) - \cos(\lambda_n x)] - [\cosh(\lambda_n l) - \cos(\lambda_n l)] [\sinh(\lambda_n x) - \sin(\lambda_n x)];$$

the  $\lambda_n$  are positive roots of the transcendental equation  $\cosh(\lambda l) \cos(\lambda l) = 1$ . The numerical values of the roots can be calculated from the formulas

$$\lambda_n = \frac{\mu_n}{l}, \quad \text{where} \quad \mu_1 = 1.875, \quad \mu_2 = 4.694, \quad \mu_n = \frac{\pi}{2}(2n - 1) \quad \text{for} \quad n \geq 3.$$

### 5.2-3. Both ends of the rod are hinged.

Boundary conditions are prescribed:

$$w = \partial_{xx} w = 0 \quad \text{at} \quad x = 0, \quad w = \partial_{xx} w = 0 \quad \text{at} \quad x = l.$$

Green's function:

$$G(x, \xi, t) = \frac{2l}{a\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(\lambda_n x) \sin(\lambda_n \xi) \sin(\lambda_n^2 at), \quad \lambda_n = \frac{\pi n}{l}.$$

### 5.2-4. One end of the rod is clamped and the other is hinged.

Boundary conditions are prescribed:

$$w = \partial_x w = 0 \quad \text{at} \quad x = 0, \quad w = \partial_{xx} w = 0 \quad \text{at} \quad x = l.$$

Green's function:

$$G(x, \xi, t) = \frac{2}{al} \sum_{n=1}^{\infty} \lambda_n^2 \frac{\varphi_n(x)\varphi_n(\xi)}{|\varphi_n'(l)\varphi_n'''(l)|} \sin(\lambda_n^2 at),$$

where

$$\varphi_n(x) = [\sinh(\lambda_n l) - \sin(\lambda_n l)] [\cosh(\lambda_n x) - \cos(\lambda_n x)] - [\cosh(\lambda_n l) - \cos(\lambda_n l)] [\sinh(\lambda_n x) - \sin(\lambda_n x)];$$

the  $\lambda_n$  are positive roots of the transcendental equation  $\tan(\lambda l) - \tanh(\lambda l) = 0$ .

### 5.2-5. One end of the rod is clamped and the other is free.

Boundary conditions are prescribed:

$$w = \partial_x w = 0 \quad \text{at } x = 0, \quad \partial_{xx} w = \partial_{xxx} w = 0 \quad \text{at } x = l.$$

Green's function:

$$G(x, \xi, t) = \frac{4}{al} \sum_{n=1}^{\infty} \frac{\varphi_n(x)\varphi_n(\xi)}{\lambda_n^2 \varphi_n^2(l)} \sin(\lambda_n^2 at),$$

where

$$\varphi_n(x) = [\sinh(\lambda_n l) + \sin(\lambda_n l)] [\cosh(\lambda_n x) - \cos(\lambda_n x)] - [\cosh(\lambda_n l) + \cos(\lambda_n l)] [\sinh(\lambda_n x) - \sin(\lambda_n x)];$$

the  $\lambda_n$  are positive roots of the transcendental equation  $\cosh(\lambda l) \cos(\lambda l) = -1$ .

### 5.2-6. One end of the rod is hinged and the other is free.

Boundary conditions are prescribed:

$$w = \partial_{xx} w = 0 \quad \text{at } x = 0, \quad \partial_{xx} w = \partial_{xxx} w = 0 \quad \text{at } x = l.$$

Green's function:

$$G(x, \xi, t) = \frac{4}{al} \sum_{n=1}^{\infty} \frac{\varphi_n(x)\varphi_n(\xi)}{\lambda_n^2 \varphi_n^2(l)} \sin(\lambda_n^2 at),$$

where

$$\varphi_n(x) = \sin(\lambda_n l) \sinh(\lambda_n x) + \sinh(\lambda_n l) \sin(\lambda_n x);$$

the  $\lambda_n$  are positive roots of the transcendental equation  $\tan(\lambda l) - \tanh(\lambda l) = 0$ .

## References

- Krylov, A. N.**, *Collected Works: III Mathematics, Pt. 2* [in Russian], Izd-vo AN SSSR, Moscow, 1949.
- Budak, B. M., Samarskii, A. A., and Tikhonov, A. N.**, *Collection of Problems on Mathematical Physics* [in Russian], Nauka, Moscow, 1980.
- Polyanin, A. D.**, *Handbook of Linear Partial Differential Equations for Engineers and Scientists*, Chapman & Hall/CRC, 2002.