5.3. Biharmonic Equation $\Delta\Delta w = 0$

The biharmonic equation is encountered in plane problems of elasticity (w is the Airy stress function). It is also used to describe slow flows of viscous incompressible fluids (w is the stream function).

In the rectangular Cartesian system of coordinates, the biharmonic operator has the form

$$\Delta\Delta \equiv \Delta^2 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}.$$

5.3-1. Particular solutions of the biharmonic equation:

$$w(x,y) = (A\cosh\beta x + B\sinh\beta x + Cx\cosh\beta x + Dx\sinh\beta x)(a\cos\beta y + b\sin\beta y),$$
$$w(x,y) = (A\cos\beta x + B\sin\beta x + Cx\cos\beta x + Dx\sin\beta x)(a\cosh\beta y + b\sinh\beta y),$$
$$w(x,y) = Ar^2\ln r + Br^2 + C\ln r + D, \qquad r = \sqrt{(x-a)^2 + (y-b)^2},$$

where A, B, C, D, E, a, b, c, and β are arbitrary constants.

5.3-2. Various representations of the general solution.

1°. Various representations of the general solution in terms of harmonic functions:

$$w(x,y) = xu_1(x,y) + u_2(x,y),$$
$$w(x,y) = yu_1(x,y) + u_2(x,y),$$
$$w(x,y) = (x^2+y^2)u_1(x,y) + u_2(x,y),$$

where u_1 and u_2 are arbitrary functions satisfying the Laplace equation $\Delta u_k = 0$ $(k = 1,\ 2)$.

2°. Complex form of representation of the general solution:

$$w(x,y) = \operatorname{Re}\bigl[\bar{z}f(z) + g(z)\bigr],$$

where $f(z)$ and $g(z)$ are arbitrary analytic functions of the complex variable $z = x + iy$; $\bar{z} = x - iy$, $i^2 = -1$. The symbol $\operatorname{Re}[A]$ stands for the real part of the complex quantity A.

5.3-3. Boundary value problems for the upper half-plane.

1°. Domain: $-\infty < x < \infty$, $0 \le y < \infty$. The desired function and its derivative along the normal are prescribed at the boundary:

$$w = 0 \quad \text{at} \quad y = 0, \qquad \partial_y w = f(x) \quad \text{at} \quad y = 0.$$

Solution:

$$w(x,y) = \int_{-\infty}^{\infty} f(\xi)G(x-\xi,y)\,d\xi, \qquad G(x,y) = \frac{1}{\pi}\frac{y^2}{x^2+y^2}.$$

2°. Domain: $-\infty < x < \infty$, $0 \le y < \infty$. The derivatives of the desired function are prescribed at the boundary:

$$\partial_x w = f(x) \quad \text{at} \quad y = 0, \qquad \partial_y w = g(x) \quad \text{at} \quad y = 0.$$

Solution:

$$w(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\left[\arctan\left(\frac{x-\xi}{y}\right) + \frac{y(x-\xi)}{(x-\xi)^2+y^2}\right]d\xi + \frac{y^2}{\pi}\int_{-\infty}^{\infty}\frac{g(\xi)\,d\xi}{(x-\xi)^2+y^2} + C,$$

where C is an arbitrary constant.

5.3-4. Boundary value problem for a circle.

Domain: $0 \le r \le a$. Boundary conditions in the polar coordinate system:

$$w = f(\varphi) \quad \text{at} \quad r = a, \qquad \partial_r w = g(\varphi) \quad \text{at} \quad r = a.$$

Solution:

$$w(r,\varphi) = \frac{1}{2\pi a}(r^2 - a^2)^2 \left[\int_0^{2\pi} \frac{[a - r\cos(\eta - \varphi)] f(\eta)\, d\eta}{[r^2 + a^2 - 2ar\cos(\eta - \varphi)]^2} - \frac{1}{2} \int_0^{2\pi} \frac{g(\eta)\, d\eta}{r^2 + a^2 - 2ar\cos(\eta - \varphi)} \right].$$

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