5.3. Biharmonic Equation $\Delta \Delta w = 0$

The biharmonic equation is encountered in plane problems of elasticity ($w$ is the Airy stress function). It is also used to describe slow flows of viscous incompressible fluids ($w$ is the stream function).

In the rectangular Cartesian system of coordinates, the biharmonic operator has the form

$$\Delta \Delta \equiv \Delta^2 = \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}.$$

5.3-1. Particular solutions of the biharmonic equation:

\begin{align*}
w(x, y) &= (A \cosh \beta x + B \sinh \beta x + C \cos \beta x + D \sin \beta x)(a \cos \beta y + b \sin \beta y), \\
w(x, y) &= (A \cos \beta x + B \sin \beta x + C \cosh \beta x + D \sinh \beta x)(a \cos \beta y + b \sin \beta y), \\
w(x, y) &= Ar^2 \ln r + Br^2 + C \ln r + D, \quad r = \sqrt{(x-a)^2 + (y-b)^2},
\end{align*}

where $A, B, C, D, E, a, b, c,$ and $\beta$ are arbitrary constants.

5.3-2. Various representations of the general solution.

1°. Various representations of the general solution in terms of harmonic functions:

\begin{align*}
w(x, y) &= xu_1(x, y) + u_2(x, y), \\
w(x, y) &= yu_1(x, y) + u_2(x, y), \\
w(x, y) &= (x^2 + y^2)u_1(x, y) + u_2(x, y),
\end{align*}

where $u_1$ and $u_2$ are arbitrary functions.

2°. Complex form of representation of the general solution:

$$w(x, y) = \text{Re} \left[ \overline{f(z)} + g(z) \right],$$

where $f(z)$ and $g(z)$ are arbitrary analytic functions of the complex variable $z = x + iy$; $\overline{z} = x - iy$, $i^2 = -1$. The symbol $\text{Re}[A]$ stands for the real part of the complex quantity $A$.

5.3-3. Boundary value problems for the upper half-plane.

1°. Domain: $-\infty < x < \infty$, $0 \leq y < \infty$. The desired function and its derivative along the normal are prescribed at the boundary:

$$w = 0 \quad \text{at} \quad y = 0, \quad \partial_y w = f(x) \quad \text{at} \quad y = 0.$$

Solution:

$$w(x, y) = \int_{-\infty}^{\infty} f(\xi) G(x-\xi, y) d\xi, \quad G(x, y) = \frac{1}{\pi} \frac{y^2}{x^2 + y^2}.$$

2°. Domain: $-\infty < x < \infty$, $0 \leq y < \infty$. The derivatives of the desired function are prescribed at the boundary:

$$\partial_x w = f(x) \quad \text{at} \quad y = 0, \quad \partial_y w = g(x) \quad \text{at} \quad y = 0.$$

Solution:

$$w(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\xi) \left[ \arctan \left( \frac{x-\xi}{y} \right) + \frac{y(x-\xi)}{(x-\xi)^2 + y^2} \right] d\xi + \frac{y^2}{\pi} \int_{-\infty}^{\infty} \frac{g(\xi)}{(x-\xi)^2 + y^2} d\xi + C,$$

where $C$ is an arbitrary constant.
5.3-4. Boundary value problem for a circle.

Domain: $0 \leq r \leq a$. Boundary conditions in the polar coordinate system:

$$w = f(\varphi) \quad \text{at} \quad r = a, \quad \partial_r w = g(\varphi) \quad \text{at} \quad r = a.$$

Solution:

$$w(r, \varphi) = \frac{1}{2\pi a} (r^2 - a^2)^2 \left[ \int_0^{2\pi} \frac{[a - r \cos(\eta - \varphi)] f(\eta) \, d\eta}{[r^2 + a^2 - 2ar \cos(\eta - \varphi)]^2} - \frac{1}{2} \int_0^{2\pi} \frac{g(\eta) \, d\eta}{r^2 + a^2 - 2ar \cos(\eta - \varphi)} \right].$$

References


