5.4. Nonhomogeneous Biharmonic Equation $\Delta\Delta w = \Phi(x,y)$

5.4-1. Domain: $-\infty < x < \infty$, $-\infty < y < \infty$.

Solution:

$$w(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi(\xi,\eta)E(x-\xi,y-\eta)\,d\xi\,d\eta, \qquad E(x,y) = \frac{1}{8\pi}(x^2+y^2)\ln\sqrt{x^2+y^2}.$$

5.4-2. Domain: $-\infty < x < \infty$, $0 \le y < \infty$. Boundary value problem.

The upper half-plane is considered. The derivatives are prescribed at the boundary:

$$\partial_x w = f(x) \quad \text{at} \quad y=0, \qquad \partial_y w = g(x) \quad \text{at} \quad y=0.$$

Solution:

$$w(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\left[\arctan\left(\frac{x-\xi}{y}\right) + \frac{y(x-\xi)}{(x-\xi)^2+y^2}\right] d\xi + \frac{y^2}{\pi}\int_{-\infty}^{\infty}\frac{g(\xi)\,d\xi}{(x-\xi)^2+y^2}$$
$$+ \frac{1}{8\pi}\int_{-\infty}^{\infty} d\xi \int_0^{\infty}\left[\frac{1}{2}(R_+^2-R_-^2) - R_-^2\ln\frac{R_+}{R_-}\right]\Phi(\xi,\eta)\,d\eta + C,$$

where C is an arbitrary constant,

$$R_+^2 = (x-\xi)^2+(y+\eta)^2, \qquad R_-^2 = (x-\xi)^2+(y-\eta)^2.$$

5.4-3. Domain: $0 \le x \le l_1$, $0 \le y \le l_2$. The sides of the plate are hinged.

A rectangle is considered. Boundary conditions are prescribed:

$$w = \partial_{xx} w = 0 \quad \text{at} \quad x=0, \qquad w = \partial_{xx} w = 0 \quad \text{at} \quad x = l_1,$$
$$w = \partial_{yy} w = 0 \quad \text{at} \quad y=0, \qquad w = \partial_{yy} w = 0 \quad \text{at} \quad y = l_2.$$

Solution:

$$w(x,y) = \int_0^{l_1}\int_0^{l_2} \Phi(\xi,\eta)G(x,y,\xi,\eta)\,d\eta\,d\xi,$$

where

$$G(x,y,\xi,\eta) = \frac{4}{l_1 l_2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(p_n^2+q_m^2)^2}\sin(p_n x)\sin(q_m y)\sin(p_n\xi)\sin(q_m\eta), \quad p_n = \frac{\pi n}{l_1}, \quad q_m = \frac{\pi m}{l_2}.$$

References

Sneddon, I., *Fourier Transformations*, McGraw-Hill, New York, 1951.

Bitsadze, A. V. and Kalinichenko, D. F., *Collection of Problems on Mathematical Physics Equations* [in Russian], Nauka, Moscow, 1985.

Polyanin, A. D., *Handbook of Linear Partial Differential Equations for Engineers and Scientists* , Chapman & Hall/CRC, 2002.

