2. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + aw - bw^3.$

Newell–Whitehead equation.

1°. Solutions with $a > 0$ and $b > 0$:

$$w(x,t) = \pm\sqrt{\frac{a}{b}}\,\frac{C_1 \exp\bigl(\frac{1}{2}\sqrt{2a}\,x\bigr) - C_2 \exp\bigl(-\frac{1}{2}\sqrt{2a}\,x\bigr)}{C_1 \exp\bigl(\frac{1}{2}\sqrt{2a}\,x\bigr) + C_2 \exp\bigl(-\frac{1}{2}\sqrt{2a}\,x\bigr) + C_3 \exp\bigl(-\frac{3}{2}at\bigr)},$$

$$w(x,t) = \pm\sqrt{\frac{a}{b}}\left[\frac{2C_1 \exp\bigl(\sqrt{2a}\,x\bigr) + C_2 \exp\bigl(\frac{1}{2}\sqrt{2a}\,x - \frac{3}{2}at\bigr)}{C_1 \exp\bigl(\sqrt{2a}\,x\bigr) + C_2 \exp\bigl(\frac{1}{2}\sqrt{2a}\,x - \frac{3}{2}at\bigr) + C_3} - 1\right],$$

where C_1, C_2, and C_3 are arbitrary constants.

2°. Solutions with $a < 0$ and $b > 0$:

$$w(x,t) = \pm\sqrt{\frac{|a|}{b}}\,\frac{\sin\bigl(\frac{1}{2}\sqrt{2|a|}\,x + C_1\bigr)}{\cos\bigl(\frac{1}{2}\sqrt{2|a|}\,x + C_1\bigr) + C_2 \exp\bigl(-\frac{3}{2}at\bigr)}.$$

3°. Solution with $a > 0$ (generalizes the first solution of Item 1°):

$$w = \bigl[C_1 \exp\bigl(\tfrac{1}{2}\sqrt{2a}\,x + \tfrac{3}{2}at\bigr) - C_2 \exp\bigl(-\tfrac{1}{2}\sqrt{2a}\,x + \tfrac{3}{2}at\bigr)\bigr]U(z),$$
$$z = C_1 \exp\bigl(\tfrac{1}{2}\sqrt{2a}\,x + \tfrac{3}{2}at\bigr) + C_2 \exp\bigl(-\tfrac{1}{2}\sqrt{2a}\,x + \tfrac{3}{2}at\bigr) + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants, and the function $U = U(z)$ is determined by the autonomous ordinary differential equation $aU''_{zz} = 2bU^3$ (whose solution can be written out in implicit form).

4°. Solution with $a < 0$ (generalizes the solution of Item 2°):

$$w = \exp\bigl(\tfrac{3}{2}at\bigr)\sin\bigl(\tfrac{1}{2}\sqrt{2|a|}\,x + C_1\bigr)V(\xi),$$
$$\xi = \exp\bigl(\tfrac{3}{2}at\bigr)\cos\bigl(\tfrac{1}{2}\sqrt{2|a|}\,x + C_1\bigr) + C_2,$$

where C_1 and C_2 are arbitrary constants, and the function $V = V(\xi)$ is determined by the autonomous ordinary differential equation $aV''_{\xi\xi} = -2bV^3$ (whose solution can be written out in implicit form).

5°. Solutions with $a = 0$ and $b > 0$:

$$w(x,t) = \pm\sqrt{\frac{2}{b}}\,\frac{2C_1 x + C_2}{C_1 x^2 + C_2 x + 6C_1 t + C_3}.$$

6°. Self-similar solution with $a = 0$:

$$w(x,t) = t^{-1/2} f(\xi), \qquad \xi = xt^{-1/2},$$

where the function $f(\xi)$ is determined by the ordinary differential equation $f''_{\xi\xi} + \frac{1}{2}\xi f'_\xi + \frac{1}{2}f - bf^3 = 0$.

7°. Solution with $a = 0$:

$$w(x,y) = xu(z), \qquad z = t + \tfrac{1}{6}x^2,$$

where the function $u(z)$ is determined by the autonomous ordinary differential equation $u''_{zz} - 9bu^3 = 0$.

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