10. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] + g(w).$

This equation occurs in nonlinear problems of heat and mass transfer with volume reaction.

1°. Traveling-wave solution:

$$w = w(z), \quad z = kx + \lambda t,$$

where k and λ are arbitrary constants, and the function $w(z)$ is determined by the autonomous ordinary differential equation $k^2[f(w)w'_z]'_z - \lambda w'_z + g(w) = 0$.

2°. Let the function $f = f(w)$ be arbitrary and let $g = g(w)$ be defined by

$$g(w) = \frac{A}{f(w)} + B,$$

where A and B are some numbers. In this case, there is a functional separable solution, which is defined implicitly by

$$\int f(w)\,dw = At - \frac{1}{2}Bx^2 + C_1 x + C_2,$$

where C_1 and C_2 are arbitrary constants.

3°. Let now $g = g(w)$ be arbitrary and let $f = f(w)$ be defined by

$$f(w) = \frac{A_1 A_2 w + B}{g(w)} + \frac{A_2 A_3}{g(w)} \int Z\,dw, \tag{1}$$

$$Z = -A_2 \int \frac{dw}{g(w)}, \tag{2}$$

where A_1, A_2, and A_3 are some numbers. Then there are generalized traveling-wave solutions of the form

$$w = w(Z), \quad Z = \frac{\pm x + C_2}{\sqrt{2A_3 t + C_1}} - \frac{A_1}{A_3} - \frac{A_2}{3A_3}(2A_3 t + C_1),$$

where the function $w(Z)$ is determined by the inversion of (2), and C_1 and C_2 are arbitrary constants.

4°. Let $g = g(w)$ be arbitrary and let $f = f(w)$ be defined by

$$f(w) = \frac{1}{g(w)}\left(A_1 w + A_3 \int Z\,dw\right) \exp\left[-A_4 \int \frac{dw}{g(w)}\right], \tag{3}$$

$$Z = \frac{1}{A_4} \exp\left[-A_4 \int \frac{dw}{g(w)}\right] - \frac{A_2}{A_4}, \tag{4}$$

where A_1, A_2, A_3, and A_4 are some numbers ($A_4 \neq 0$). In this case, there are generalized traveling-wave solutions of the form

$$w = w(Z), \quad Z = \varphi(t)x + \psi(t),$$

where the function $w(Z)$ is determined by the inversion of (4),

$$\varphi(t) = \pm\left(C_1 e^{2A_4 t} - \frac{A_3}{A_4}\right)^{-1/2}, \quad \psi(t) = -\varphi(t)\left[A_1 \int \varphi(t)\,dt + A_2 \int \frac{dt}{\varphi(t)} + C_2\right],$$

and C_1 and C_2 are arbitrary constants.

5°. Let the functions $f(w)$ and $g(w)$ be as follows:

$$f(w) = \varphi'(w), \quad g(w) = \frac{a\varphi(w) + b}{\varphi'(w)} + c[a\varphi(w) + b],$$

where $\varphi(w)$ is an arbitrary function and a, b, and c are any numbers (the prime denotes a derivative with respect to w). Then there are functional separable solutions defined implicitly by

$$\varphi(w) = e^{at}\big[C_1\cos(x\sqrt{ac}\,) + C_2\sin(x\sqrt{ac}\,)\big] - \frac{b}{a} \qquad \text{if } ac > 0,$$

$$\varphi(w) = e^{at}\big[C_1\cosh(x\sqrt{-ac}\,) + C_2\sinh(x\sqrt{-ac}\,)\big] - \frac{b}{a} \qquad \text{if } ac < 0.$$

6°. Let $f(w)$ and $g(w)$ be as follows:

$$f(w) = w\varphi'_w(w), \qquad g(w) = a\left[w + 2\frac{\varphi(w)}{\varphi'_w(w)}\right],$$

where $\varphi(w)$ is an arbitrary function and a is any number. Then there are functional separable solutions defined implicitly by

$$\varphi(w) = C_1e^{2at} - \tfrac{1}{2}a(x + C_2)^2.$$

7°. Let $f(w)$ and $g(w)$ be defined by the formulas

$$f(w) = A\frac{V(z)}{V'_z(z)}, \qquad g(w) = B\big[2z^{-1/2}V'_z(z) + Bz^{-3/2}V(z)\big],$$

where $V(z)$ is an arbitrary function of z, A and B are arbitrary constants ($AB \neq 0$), and the function $z = z(w)$ is determined implicitly by

$$w = \int z^{-1/2}V'_z(z)\,dz + C_1, \tag{5}$$

with C_1 being an arbitrary constant. Then, there is a functional separable solution of the form (5) where

$$z = -\frac{(x + C_3)^2}{4At + C_2} + 2Bt + \frac{BC_2}{2A},$$

C_2 and C_3 are arbitrary constants.

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