1. $i\dfrac{\partial w}{\partial t} + \dfrac{\partial^2 w}{\partial x^2} + k|w|^2 w = 0.$

Schrodinger (Schrödinger) equation with a cubic nonlinearity. Here, w is a complex functions of real variables x and t; k is a real number, $i^2 = -1$. This equation occurs in various chapters of physics, including nonlinear optics, superconductivity, and plasma physics.

1°. Solutions:

$$w(x,t) = C_1 \exp\bigl\{i\,[C_2 x + (kC_1^2 - C_2^2)t + C_3]\bigr\},$$
$$w(x,t) = \pm C_1\sqrt{\frac{2}{k}}\,\frac{\exp[i(C_1^2 t + C_2)]}{\cosh(C_1 x + C_3)},$$
$$w(x,t) = \pm A\sqrt{\frac{2}{k}}\,\frac{\exp[iBx + i(A^2 - B^2)t + iC_1]}{\cosh(Ax - 2ABt + C_2)},$$
$$w(x,t) = \frac{C_1}{\sqrt{t}}\exp\left[i\frac{(x+C_2)^2}{4t} + i(kC_1^2 \ln t + C_3)\right],$$

where A, B, C_1, C_2, and C_3 are arbitrary real constants. The second and third solutions are valid for $k > 0$. The third solution describes the motion of a soliton in a rapidly decaying case.

2°. N-soliton solutions for $k > 0$:

$$w(x,t) = \sqrt{\frac{2}{k}}\,\frac{\det \mathbf{R}(x,t)}{\det \mathbf{M}(x,t)}.$$

Here, $\mathbf{M}(x,t)$ is an $N \times N$ matrix with entries

$$M_{n,k}(x,t) = \frac{1 + \bar{g}_n(x,t) g_n(x,t)}{\bar{\lambda}_n - \lambda_k}, \qquad g_n(x,t) = \gamma_n e^{i(\lambda_n x - \lambda_n^2 t)}, \qquad n,\ k = 1,\ \ldots,\ N,$$

where the λ_n and γ_n are arbitrary complex numbers that satisfy the constraints $\operatorname{Im}\lambda_n > 0$ ($\lambda_n \neq \lambda_k$ if $n \neq k$) and $\gamma_n \neq 0$; the bar over a symbol denotes the complex conjugate. The square matrix $\mathbf{R}(x,t)$ is of order $N+1$; it is obtained by augmenting $\mathbf{M}(x,t)$ with a column on the right and a row at the bottom. The entries of $\mathbf{R}$ are defined as

$$\begin{aligned}
&R_{n,k}(x,t) = M_{n,k}(x,t) && \text{for} \quad n,k = 1,\ \ldots,\ N && \text{(bulk of the matrix)},\\
&R_{n,N+1}(x,t) = g_n(x,t) && \text{for} \quad n = 1,\ \ldots,\ N && \text{(rightmost column)},\\
&R_{N+1,n}(x,t) = 1 && \text{for} \quad n = 1,\ \ldots,\ N && \text{(bottom row)},\\
&R_{N+1,N+1}(x,t) = 0 && && \text{(lower right diagonal entry)}.
\end{aligned}$$

The above solution can be represented, for $t \to \pm\infty$, as the sum of N single-soliton solutions.

4°. For other exact solutions, see the Schrodinger equation with a power-law nonlinearity with $n = 1$ and the nonlinear Schrodinger equation of general form with $f(u) = ku^2$.

5°. The Schrodinger equation with a cubic nonlinearity is integrable by the inverse scattering method.

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