1. \( \frac{\partial w}{\partial t} + \frac{\partial^2 w}{\partial x^2} + k|w|^2 w = 0. \)

**Schrodinger (Schrödinger) equation with a cubic nonlinearity.** Here, \( w \) is a complex functions of real variables \( x \) and \( t \); \( k \) is a real number, \( i^2 = -1 \). This equation occurs in various chapters of physics, including nonlinear optics, superconductivity, and plasma physics.

1°. Solutions:

\[
\begin{align*}
&w(x, t) = C_1 \exp \left\{ i \left[ C_2 x + (kC_1^2 - C_2^2)t + C_3 \right] \right\}, \\
&w(x, t) = \pm C_1 \sqrt{\frac{2}{k}} \exp \left\{ i \left( C_1^2 t + C_2 \right) \right\}, \\
&w(x, t) = \pm A \sqrt{\frac{2}{k}} \exp \left\{ \frac{iBx + i(A^2 - B^2)t + iC_1}{\cosh(Ax - 2ABt + C_2)} \right\}, \\
&w(x, t) = \frac{C_1}{\sqrt{t}} \exp \left\{ \frac{i(x + C_1)^2}{4t} + i(kC_1^2 \ln t + C_3) \right\},
\end{align*}
\]

where \( A, B, C_1, C_2, \) and \( C_3 \) are arbitrary real constants. The second and third solutions are valid for \( k > 0 \). The third solution describes the motion of a soliton in a rapidly decaying case.

2°. \( N \)-soliton solutions for \( k > 0 \):

\[
w(x, t) = \sqrt{\frac{2}{k}} \det R(x, t).
\]

Here, \( R(x, t) \) is an \( N \times N \) matrix with entries

\[
M_{n,k}(x, t) = \frac{1 + g_n(x, t)g_n(x, t)}{\lambda_n - \lambda_k}, \quad g_n(x, t) = \gamma_n e^{i(\lambda_n x - \lambda_n^2 t)} , \quad n, k = 1, \ldots, N,
\]

where the \( \lambda_n \) and \( \gamma_n \) are arbitrary complex numbers that satisfy the constraints \( \text{Im} \lambda_n > 0 \) (\( \lambda_n \neq \lambda_k \) if \( n \neq k \)) and \( \gamma_n \neq 0 \); the bar over a symbol denotes the complex conjugate. The square matrix \( R(x, t) \) is of order \( N + 1 \); it is obtained by augmenting \( M(x, t) \) with a column on the right and a row at the bottom. The entries of \( R \) are defined as

\[
\begin{align*}
R_{n,k}(x, t) &= M_{n,k}(x, t) \quad \text{for} \quad n, k = 1, \ldots, N \quad \text{(bulk of the matrix)}, \\
R_{n,N+1}(x, t) = g_n(x, t) \quad \text{for} \quad n = 1, \ldots, N \quad \text{(rightmost column)}, \\
R_{N+1,n}(x, t) &= 1 \quad \text{for} \quad n = 1, \ldots, N \quad \text{(bottom row)}, \\
R_{N+1, N+1}(x, t) &= 0 \quad \text{(lower right diagonal entry)}.
\end{align*}
\]

The above solution can be represented, for \( t \to \pm \infty \), as the sum of \( N \) single-soliton solutions.

4°. For other exact solutions, see the [Schrodinger equation with a power-law nonlinearity] with \( n = 1 \) and the [nonlinear Schrodinger equation of general form] with \( f(u) = ku^2 \).

5°. The Schrodinger equation with a cubic nonlinearity is integrable by the inverse scattering method.

**References**


Schroedinger Equation with a Cubic Nonlinearity

Copyright © 2004 Andrei D. Polyanin