7. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + f(w).$

Nonlinear Klein–Gordon equation.

1°. Suppose $w = w(x,t)$ is a solution of the nonlinear Klein–Gordon equation. Then the functions

$$w_1 = w(\pm x + C_1, \pm t + C_2),$$
$$w_2 = w(x\cosh\beta + t\sinh\beta,\ t\cosh\beta + x\sinh\beta),$$

where C_1, C_2, and β are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

2°. Traveling-wave solution in implicit form:

$$\int\left[C_1 + \frac{2}{\lambda^2 - k^2}\int f(w)\,dw\right]^{-1/2} dw = kx + \lambda t + C_2,$$

where C_1, C_2, k, and λ are arbitrary constants.

3°. Functional separable solution:

$$w = w(\xi), \qquad \xi = \tfrac{1}{4}(t + C_1)^2 - \tfrac{1}{4}(x + C_2)^2,$$

where C_1 and C_2 are arbitrary constants, and the function $w = w(\xi)$ is determined by the ordinary differential equation $\xi w''_{\xi\xi} + w'_\xi - f(w) = 0$.

See also special cases of the nonlinear Klein–Gordon equation:

- Klein–Gordon equation with a power-law nonlinearity - 1 ,
- Klein–Gordon equation with a power-law nonlinearity - 2 ,
- modified Liouville equation ,
- Klein–Gordon equation with a exponential nonlinearity ,
- sinh-Gordon equation ,
- sine-Gordon equation .

References

Clarkson, P. A., McLeod, J. B., Olver, P. J., and Ramani, R., Integrability of Klein–Gordon equations, *SIAM J. Math. Anal.*, Vol. 17, pp. 798–802, 1986.

Matsuno, Y., Exact solutions for the nonlinear Klein–Gordon and Liouville equations in for dimensional Euclidean space, *J. Math. Physics*, Vol. 28, No. 10, pp. 2317–2322, 1987.

Grundland, A. M. and Infeld, E., A family of non-linear Klein-Gordon equations and their solutions, *J. Math. Phys.*, Vol. 33, pp. 2498–2503, 1992.

Zhdanov, R. Z., Separation of variables in the non-linear wave equation, *J. Phys. A*, Vol. 27, pp. L291–L297, 1994.

Polyanin, A. D. and Zaitsev, V. F., *Handbook of Nonlinear Partial Differential Equations* , Chapman & Hall/CRC, Boca Raton, 2004.

