3. $\dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right]+\dfrac{\partial}{\partial y}\left[g(w)\dfrac{\partial w}{\partial y}\right]=0.$

This is a *stationary anisotropic heat (diffusion) equation*.

1°. Traveling-wave solution in implicit form:

$$\int\left[A^2 f(w)+B^2 g(w)\right]dw=C_1(Ax+By)+C_2,$$

where A, B, C_1, and C_2 are arbitrary constants.

2°. Self-similar solution:

$$w=w(\zeta),\qquad \zeta=\frac{x+A}{y+B},$$

where the function $w(\zeta)$ is determined by the ordinary differential equation

$$[f(w)w'_\zeta]'_\zeta+[\zeta^2 g(w)w'_\zeta]'_\zeta=0. \tag{1}$$

Integrating (1) and taking w to be the independent variable, one obtains the Riccati equation $C\zeta'_w=g(w)\zeta^2+f(w)$, where C is an arbitrary constant.

3°. The original equation can be represented as the system of the equations

$$f(w)\frac{\partial w}{\partial x}=\frac{\partial v}{\partial y},\qquad -g(w)\frac{\partial w}{\partial y}=\frac{\partial v}{\partial x}. \tag{2}$$

The hodograph transformation

$$x=x(w,v),\qquad y=y(w,v),$$

where w, v are treated as the independent variables and x, y as the dependent ones, brings (2) to the linear system

$$f(w)\frac{\partial y}{\partial v}=\frac{\partial x}{\partial w},\qquad -g(w)\frac{\partial x}{\partial v}=\frac{\partial y}{\partial w}. \tag{3}$$

Eliminating y yields the following linear equation for $x=x(w,v)$:

$$\frac{\partial}{\partial w}\left[\frac{1}{f(w)}\frac{\partial x}{\partial w}\right]+g(w)\frac{\partial^2 x}{\partial v^2}=0.$$

Likewise, we can obtain another linear equation for $y=y(w,v)$ from system (3).

Reference

Polyanin, A. D. and Zaitsev, V. F., *Handbook of Nonlinear Partial Differential Equations*, Chapman & Hall/CRC, Boca Raton, 2004.

