Korteweg–de Vries equation. It is used in many sections of nonlinear mechanics and physics.

1°. Suppose \( w(x, t) \) is a solution of the Korteweg–de Vries equation. Then the function

\[
\begin{align*}
    w_1 &= C_1^2 w(C_1 x + 6C_1 C_2 t + C_5, C_1^2 t + C_4) + C_2,
\end{align*}
\]

where \( C_1, \ldots, C_4 \) are arbitrary constants, is also a solution of the equation.

2°. One-soliton solution:

\[
\begin{align*}
    w(x, t) &= -\frac{a}{2 \cosh^2 \left[ \frac{1}{2} \sqrt{a} (x - at - b) \right]},
\end{align*}
\]

where \( a \) and \( b \) are arbitrary constants.

3°. Two-soliton solution:

\[
\begin{align*}
    w(x, t) &= -2 \frac{\partial^2}{\partial x^2} \ln \left( 1 + B_1 e^{\theta_1} + B_2 e^{\theta_2} + AB_1 B_2 e^{\theta_1 + \theta_2} \right),
\end{align*}
\]

\[
\begin{align*}
    \theta_1 &= a_1 x - a_1^2 t, \quad \theta_2 = a_2 x - a_2^2 t, \quad A = \left( \frac{a_1 - a_2}{a_1 + a_2} \right)^2.
\end{align*}
\]

where \( B_1, B_2, a_1, \) and \( a_2 \) are arbitrary constants.

4°. \( N \)-soliton solution:

\[
\begin{align*}
    w(x, t) &= -2 \frac{\partial^2}{\partial x^2} \left\{ \ln \det \left[ \mathbf{I} + \mathbf{C}(x, t) \right] \right\}.
\end{align*}
\]

Here, \( \mathbf{I} \) is the \( N \times N \) identity matrix and \( \mathbf{C}(x, t) \) the \( N \times N \) symmetric matrix with entries

\[
C_{mn}(x, t) = \frac{\sqrt{\rho_m(t)\rho_n(t)}}{\rho_m + \rho_n} \exp \left[ -\left( \rho_m + \rho_n \right)x \right],
\]

where the normalizing factors \( \rho_n(t) \) are given by

\[
\rho_n(t) = \rho_n(0) \exp(8\rho_n^2 t), \quad n = 1, 2, \ldots, N.
\]

The solution involves \( 2^N \) arbitrary constants \( \rho_n \) and \( \rho_n(0) \).

The above solution can be represented, for \( t \to \pm \infty \), as the sum of \( N \) single-soliton solutions.

5°. “One soliton + one pole” solution:

\[
\begin{align*}
    w(x, t) &= -2p^2 \left[ \cosh^2(pz) - (1 + px)^{-2} \tanh^2(pz) \right] \left[ 1 - (1 + px)^{-1} \tanh(pz) \right]^{-2}, \quad z = x - 4p^2 t - c,
\end{align*}
\]

where \( p \) and \( c \) are arbitrary constants.

6°. Rational solutions (algebraic solitons):

\[
\begin{align*}
    w(x, t) &= \frac{6x(x^3 - 24t)}{(x^3 + 12t)^2},
\end{align*}
\]

\[
\begin{align*}
    w(x, t) &= -2 \frac{\partial^2}{\partial x^2} \ln(x^6 + 60x^3 t - 720t^2).
\end{align*}
\]

7°. There is a self-similar solution of the form \( w = t^{-2/3} U(z) \), where \( z = t^{-1/3} x \).
8°. Solution:

$$w(x, t) = 2\varphi(z) + 2C_1 t, \quad z = x + 6C_1 t^2 + C_2 t,$$

where $C_1$ and $C_2$ are arbitrary constants, and the function $\varphi(z)$ is determined by the second-order ordinary differential equation $\varphi''_{zz} = 6\varphi^2 - C_2 \varphi - C_1 z + C_3$.

9°. The Korteweg–de Vries equation is solved by the inverse scattering method. Any rapidly decreasing function $F = F(x, y, t)$ as $x \to +\infty$ that satisfies simultaneously the two linear equations

$$\nabla^2 F - \frac{\partial^2 F}{\partial y^2} = 0, \quad \frac{\partial F}{\partial t} + \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^3 F = 0$$

generates a solution of the Korteweg–de Vries equation in the form

$$w = -2 \frac{d}{dx} K(x, x; t),$$

where $K(x, y; t)$ is a solution of the linear Gel'fand–Levitan–Marchenko integral equation

$$K(x, y; t) + F(x, y; t) + \int_{x}^{\infty} K(x, z; t) F(z, y; t) \, dz = 0.$$ 

Time $t$ appears in this equation as a parameter.

See also:
- cylindrical Korteweg–de Vries equation,
- modified Korteweg–de Vries equation,
- generalized Korteweg–de Vries equation.

References


