1. $\dfrac{\partial w}{\partial t} + \dfrac{\partial^3 w}{\partial x^3} - 6w\dfrac{\partial w}{\partial x} = 0.$

Korteweg–de Vries equation. It is used in many sections of nonlinear mechanics and physics.

1°. Suppose $w(x,t)$ is a solution of the Korteweg–de Vries equation. Then the function

$$w_1 = C_1^2 w(C_1 x + 6C_1C_2 t + C_3,\, C_1^3 t + C_4) + C_2,$$

where $C_1, \dots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. One-soliton solution:

$$w(x,t) = -\frac{a}{2\cosh^2\left[\frac{1}{2}\sqrt{a}\,(x - at - b)\right]},$$

where a and b are arbitrary constants.

3°. Two-soliton solution:

$$w(x,t) = -2\frac{\partial^2}{\partial x^2}\ln\left(1 + B_1e^{\theta_1} + B_2e^{\theta_2} + AB_1B_2e^{\theta_1+\theta_2}\right),$$
$$\theta_1 = a_1x - a_1^3t, \quad \theta_2 = a_2x - a_2^3t, \quad A = \left(\frac{a_1 - a_2}{a_1 + a_2}\right)^2,$$

where B_1, B_2, a_1, and a_2 are arbitrary constants.

4°. N-soliton solution:

$$w(x,t) = -2\frac{\partial^2}{\partial x^2}\left\{\ln\det\left[\mathbf{I} + \mathbf{C}(\mathbf{x},\mathbf{t})\right]\right\}.$$

Here, $\mathbf{I}$ is the $N\times N$ identity matrix and $\mathbf{C}(x,t)$ the $N\times N$ symmetric matrix with entries

$$C_{mn}(x,t) = \frac{\sqrt{\rho_m(t)\rho_n(t)}}{p_m + p_n}\exp\left[-(p_m + p_n)x\right],$$

where the normalizing factors $\rho_n(t)$ are given by

$$\rho_n(t) = \rho_n(0)\exp\left(8p_n^3t\right), \quad n = 1, 2, \dots, N.$$

The solution involves $2N$ arbitrary constants p_n and $\rho_n(0)$.

The above solution can be represented, for $t \to \pm\infty$, as the sum of N single-soliton solutions.

5°. "One soliton + one pole" solution:

$$w(x,t) = -2p^2\left[\cosh^{-2}(pz) - (1+px)^{-2}\tanh^2(pz)\right]\left[1 - (1+px)^{-1}\tanh(pz)\right]^{-2}, \quad z = x - 4p^2t - c,$$

where p and c are arbitrary constants.

6°. Rational solutions (algebraic solitons):

$$w(x,t) = \frac{6x(x^3 - 24t)}{(x^3 + 12t)^2},$$
$$w(x,t) = -2\frac{\partial^2}{\partial x^2}\ln(x^6 + 60x^3t - 720t^2).$$

7°. There is a self-similar solution of the form $w = t^{-2/3}U(z)$, where $z = t^{-1/3}x$.

8°. Solution:

$$w(x,t) = 2\varphi(z) + 2C_1 t, \qquad z = x + 6C_1 t^2 + C_2 t,$$

where C_1 and C_2 are arbitrary constants, and the function $\varphi(z)$ is determined by the second-order ordinary differential equation $\varphi''_{zz} = 6\varphi^2 - C_2\varphi - C_1 z + C_3$.

9°. The Korteweg–de Vries equation is solved by the inverse scattering method. Any rapidly decreasing function $F = F(x,y;t)$ as $x \to +\infty$ that satisfies simultaneously the two linear equations

$$\frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 F}{\partial y^2} = 0, \qquad \frac{\partial F}{\partial t} + \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^3 F = 0$$

generates a solution of the Korteweg–de Vries equation in the form

$$w = -2\frac{d}{dx}K(x,x;t),$$

where $K(x,y;t)$ is a solution of the linear Gel'fand–Levitan–Marchenko integral equation

$$K(x,y;t) + F(x,y;t) + \int_x^\infty K(x,z;t)F(z,y;t)\,dz = 0.$$

Time t appears in this equation as a parameter.

See also:

- cylindrical Korteweg–de Vries equation ,
- modified Korteweg–de Vries equation ,
- generalized Korteweg–de Vries equation .

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