3. $\dfrac{\partial w}{\partial t} + \dfrac{\partial^3 w}{\partial x^3} + 6\sigma w^2 \dfrac{\partial w}{\partial x} = 0.$

Modified Korteweg–de Vries equation.

1°. One-soliton solution for $\sigma = 1$:

$$w(x,t) = a + \frac{k^2}{\sqrt{4a^2+k^2}\,\cosh z + 2a}, \qquad z = kx - (6a^2k + k^3)t + b,$$

where a, b, and k are arbitrary constants.

2°. Two-soliton solution for $\sigma = 1$:

$$w(x,t) = 2\frac{a_1 e^{\theta_1} + a_2 e^{\theta_2} + A a_2 e^{2\theta_1+\theta_2} + A a_1 e^{\theta_1+2\theta_2}}{1 + e^{2\theta_1} + e^{2\theta_2} + 2(1-A)e^{\theta_1+\theta_2} + Ae^{2(\theta_1+\theta_2)}},$$

$$\theta_1 = a_1 x - a_1^3 t + b_1, \quad \theta_2 = a_2 x - a_2^3 t + b_2, \quad A = \left(\frac{a_1 - a_2}{a_1 + a_2}\right)^2,$$

where a_1, a_2, b_1, and b_2 are arbitrary constants.

3°. Rational solutions (algebraic solitons) for $\sigma = 1$:

$$w(x,t) = a - \frac{4a}{4a^2z^2+1}, \qquad z = x - 6a^2t,$$

$$w(x,t) = a - \frac{12a\left(z^4 + \frac{3}{2}a^{-2}z^2 - \frac{3}{16}a^{-4} - 24tz\right)}{4a^2\left(z^3 + 12t - \frac{3}{4}a^{-2}z\right)^2 + 3\left(z^2 + \frac{1}{4}a^{-2}\right)^2},$$

where a is an arbitrary constant.

4°. There is a self-similar solution of the form $w = t^{-1/3}U(z)$, where $z = t^{-1/3}x$.

5°. The modified Korteweg–de Vries equation is solved by the inverse scattering method.

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