5. $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x \partial y} - \dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2} = \nu \dfrac{\partial^3 w}{\partial y^3}.$

This is an ***equation of a steady-state laminar boundary layer*** on a flat plate (it is obtained from the boundary layer equations by introducing the stream function w, see Remark).

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w\bigl(C_2 x + C_3,\, C_1 C_2 y + \varphi(x)\bigr) + C_4,$$

where $\varphi(x)$ is an arbitrary function and $C_1, \ldots, C_5$ are arbitrary constants, is also a solution of the equation.

2°. Solutions involving arbitrary function:

$$\begin{aligned}
w(x,y) &= C_1 y + \varphi(x),\\
w(x,y) &= C_1 y^2 + \varphi(x) y + \frac{1}{4C_1}\varphi^2(x) + C_2,\\
w(x,y) &= \frac{6\nu x + C_1}{y + \varphi(x)} + \frac{C_2}{[y+\varphi(x)]^2} + C_3,\\
w(x,y) &= \varphi(x)\exp(-C_1 y) + \nu C_1 x + C_2,\\
w(x,y) &= C_1 \exp\bigl[-C_2 y - C_2\varphi(x)\bigr] + C_3 y + C_3\varphi(x) + \nu C_2 x + C_4,\\
w(x,y) &= 6\nu C_1 x^{1/3}\tanh\xi + C_2, \qquad \xi = C_1\frac{y}{x^{2/3}} + \varphi(x),\\
w(x,y) &= -6\nu C_1 x^{1/3}\tan\xi + C_2, \qquad \xi = C_1\frac{y}{x^{2/3}} + \varphi(x),
\end{aligned}$$

where $C_1, \ldots, C_4$ are arbitrary constants and $\varphi(x)$ is an arbitrary function. The first and second solutions are degenerate solutions; its are independent of ν and correspond to inviscid fluid flows.

3°. Table 5 lists invariant solutions to the hydrodynamic boundary layer equation. Solution 1 is expressed in additive separable form, solution 2 is in multiplicative separable form, solution 3 is self-similar, and solution 4 is generalized self-similar. Solution 5 degenerates at $a = 0$ into a self-similar solution (see solution 3 with $\lambda = -1$). Equations 3–5 for F are autonomous and generalized homogeneous; hence, their order can be reduced by two.

TABLE
Invariant solutions to the hydrodynamic boundary layer equation (the additive constant is omitted)

No.	Solution structure	Function F or equation for F	Remarks
1	$w = F(y) + \nu\lambda x$	$F(y) = \begin{cases} C_1\exp(-\lambda y) + C_2 y & \text{if } \lambda \neq 0,\\ C_1 y^2 + C_2 y & \text{if } \lambda = 0 \end{cases}$	λ is any
2	$w = F(x) y^{-1}$	$F(x) = 6\nu x + C_1$	—
3	$w = x^{\lambda+1}F(z),\ z = x^{\lambda} y$	$(2\lambda+1)(F'_z)^2 - (\lambda+1)FF''_{zz} = \nu F'''_{zzz}$	λ is any
4	$w = e^{\lambda x}F(z),\ z = e^{\lambda x} y$	$2\lambda(F'_z)^2 - \lambda FF''_{zz} = \nu F'''_{zzz}$	λ is any
5	$w = F(z) + a\ln\lvert x\rvert,\ z = y/x$	$-(F'_z)^2 - aF''_{zz} = \nu F'''_{zzz}$	a is any

4°. Generalized separable solution linear in x:

$$w(x,y) = xf(y) + g(y), \tag{1}$$

where the functions $f = f(y)$ and $g = g(y)$ are determined by the autonomous system of ordinary differential equations

$$(f'_y)^2 - ff''_{yy} = \nu f'''_{yyy}, \tag{2}$$
$$f'_y g'_y - fg''_{yy} = \nu g'''_{yyy}. \tag{3}$$

Equation (2) has the following particular solutions:

$$f = 6\nu(y+C)^{-1},$$
$$f = Ce^{\lambda y} - \lambda\nu,$$

where C and λ are arbitrary constants.

Let $f = f(y)$ is a solution of equation (2) ($f \not\equiv \text{const}$). Then, the corresponding general solution of equation (3) can be written out in the form

$$g(y) = C_1 + C_2 f + C_3\left(f\int \psi\,dy - \int f\psi\,dy\right), \quad \text{where} \quad \psi = \frac{1}{(f'_y)^2}\exp\left(-\frac{1}{\nu}\int f\,dy\right).$$

Remark. The system of hydrodynamic boundary layer equations

$$u_1\frac{\partial u_1}{\partial x} + u_2\frac{\partial u_1}{\partial y} = \nu\frac{\partial^2 u_1}{\partial y^2},$$
$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0,$$

where u_1 and u_2 are the longitudinal and normal components of the fluid velocity, respectively, is reduced to the equation in question by the introduction of a stream function w such that $u_1 = \frac{\partial w}{\partial y}$ and $u_2 = -\frac{\partial w}{\partial x}$.

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