6. $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x \partial y} - \dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2} = \nu \dfrac{\partial^3 w}{\partial y^3} + f(x).$

This is a ***hydrodynamic boundary layer equation with pressure gradient*** (it is obtained from the boundary layer equations by introducing the stream function w.

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the functions

$$w_1 = \pm w(x, \pm y + \varphi(x)) + C,$$

where $\varphi(x)$ is an arbitrary function and C is an arbitrary constant, are also solutions of the equation.

2°. Degenerate solutions (linear and quadratic in y) for arbitrary $f(x)$:

$$w(x,y) = \pm y\left[2\int f(x)\,dx + C_1\right]^{1/2} + \varphi(x),$$

$$w(x,y) = C_1 y^2 + \varphi(x) y + \frac{1}{4C_1}\left[\varphi^2(x) - 2\int f(x)\,dx\right] + C_2,$$

where $\varphi(x)$ is an arbitrary function, and C_1 and C_2 are arbitrary constants. These solutions are independent of ν and correspond to inviscid fluid flows.

3°. Table lists invariant solutions to the boundary layer equation with pressure gradient.

TABLE
Invariant solutions to the hydrodynamic boundary layer equation
with pressure gradient (a, k, m, and β are arbitrary constants)

No.	Function $f(x)$	Form of solution $w = w(x,y)$	Function u or equation for u
1	$f(x) = 0$	See equation 5.1.5	See equation 5.1.5
2	$f(x) = ax^m$	$w = x^{\frac{m+3}{4}} u(z),\ z = x^{\frac{m-1}{4}} y$	$\frac{m+1}{2}(u'_z)^2 - \frac{m+3}{4} u u''_{zz} = \nu u'''_{zzz} + a$
3	$f(x) = ae^{\beta x}$	$w = e^{\frac{1}{4}\beta x} u(z),\ z = e^{\frac{1}{4}\beta x} y$	$\frac{1}{2}\beta (u'_z)^2 - \frac{1}{4}\beta u u''_{zz} = \nu u'''_{zzz} + a$
4	$f(x) = a$	$w = kx + u(y)$	$u(y) = \begin{cases} C_1 \exp\left(-\frac{k}{\nu} y\right) - \frac{a}{2k} y^2 + C_2 y & \text{if } k \neq 0, \\ -\frac{a}{6\nu} y^3 + C_2 y^2 + C_1 y & \text{if } k = 0 \end{cases}$
5	$f(x) = ax^{-3}$	$w = k \ln\lvert x\rvert + u(z),\ z = y/x$	$-(u'_z)^2 - k u''_{zz} = \nu u'''_{zzz} + a$

4°. Generalized separable solution for $f(x) = ax + b$:

$$w(x,y) = xF(y) + G(y),$$

where the functions $F = F(y)$ and $G = G(y)$ are determined by the system of ordinary differential equations

$$(F'_y)^2 - FF''_{yy} = \nu F'''_{yyy} + a, \qquad F'_y G'_y - FG''_{yy} = \nu G'''_{yyy} + b.$$

5°. Solutions for $f(x) = -ax^{-5/3}$:

$$w(x,y) = \frac{6\nu x}{y + \varphi(x)} \pm \frac{\sqrt{3a}}{x^{1/3}}[y + \varphi(x)],$$

where $\varphi(x)$ is an arbitrary function.

6°. Solutions for $f(x) = ax^{-1/3} - bx^{-5/3}$:

$$w(x,y) = \pm\sqrt{3b}\,z + x^{2/3}\theta(z), \qquad z = yx^{-1/3},$$

where the function $\theta = \theta(z)$ is determined by the ordinary differential equation $\frac{1}{3}(\theta'_z)^2 - \frac{2}{3}\theta\theta''_{zz} = \nu\theta'''_{zzz} + a$.

7°. Generalized separable solution for $f(x) = ae^{\beta x}$:

$$w(x,y) = \varphi(x)e^{\lambda y} - \frac{a}{2\beta\lambda^2\varphi(x)}e^{\beta x - \lambda y} - \nu\lambda x + \frac{2\nu\lambda^2}{\beta}y + \frac{2\nu\lambda}{\beta}\ln|\varphi(x)|,$$

where $\varphi(x)$ is an arbitrary function and λ is an arbitrary constant.

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