1. $\dfrac{\partial^2 w}{\partial t^2} + \dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial^4 w}{\partial x^4} = 0.$

Boussinesq equation. This equation arises in hydrodynamics and some physical applications.

1°. Suppose $w(x,t)$ is a solution of the Boussinesq equation in question. Then the function

$$w_1 = C_1^2 w(C_1 x + C_2, \pm C_1^2 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Solutions:

$$w(x,t) = 2C_1 x - 2C_1^2 t^2 + C_2 t + C_3,$$
$$w(x,t) = (C_1 t + C_2)x - \frac{1}{12C_1^2}(C_1 t + C_2)^4 + C_3 t + C_4,$$
$$w(x,t) = -\frac{(x+C_1)^2}{(t+C_2)^2} + \frac{C_3}{t+C_2} + C_4(t+C_2)^2,$$
$$w(x,t) = -\frac{x^2}{t^2} + C_1 t^3 x - \frac{C_1^2}{54}t^8 + C_2 t^2 + \frac{C_4}{t},$$
$$w(x,t) = -\frac{(x+C_1)^2}{(t+C_2)^2} - \frac{12}{(x+C_1)^2},$$
$$w(x,t) = -3\lambda^2 \cos^{-2}\left[\tfrac{1}{2}\lambda(x \pm \lambda t) + C_1\right],$$

where $C_1, \ldots, C_4$ and λ are arbitrary constants.

3°. Traveling-wave solution (generalizes the last solution of Item 1°):

$$w(x,t) = w(\zeta), \qquad \zeta = x + \lambda t,$$

where the function $w(\zeta)$ is determined by the second-order ordinary differential equation $w''_{\zeta\zeta} + w^2 + 2\lambda^2 w + C_1\zeta + C_2 = 0$.

4°. Self-similar solution:

$$w(x,t) = t^{-1}u(z), \qquad z = xt^{-1/2},$$

where the function $u = u(z)$ is determined by the ordinary differential equation $u''''_{zzzz} + (uu'_z)'_z + \frac{1}{4}z^2 u''_{zz} + \frac{7}{4}zu'_z + 2u = 0$.

5°. There are exact solutions of the following forms:

$$w(x,t) = (x+C)^2 F(t) - 12(x+C)^{-2};$$
$$w(x,t) = G(\xi) - 4C_1^2 t^2 - 4C_1 C_2 t, \qquad \xi = x - C_1 t^2 - C_2 t;$$
$$w(x,t) = \frac{1}{t}H(\eta) - \frac{1}{4}\left(\frac{x}{t} + Ct\right)^2, \qquad \eta = \frac{x}{\sqrt{t}} - \frac{1}{3}Ct^{3/2};$$
$$w(x,t) = (a_1 t + a_0)^2 U(\zeta) - \left(\frac{a_1 x + b_1}{a_1 t + a_0}\right)^2, \qquad \zeta = x(a_1 t + a_0) + b_1 t + b_0,$$

where C, C_1, C_2, a_1, a_0, b_1, and b_0 are arbitrary constants,

6°. The Boussinesq equation is solved by the inverse scattering method. Any rapidly decaying function $F = F(x,y;t)$ as $x \to +\infty$ and satisfying simultaneously the two linear equations

$$\frac{1}{\sqrt{3}}\frac{\partial F}{\partial t} + \frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 F}{\partial y^2} = 0, \qquad \frac{\partial^3 F}{\partial x^3} + \frac{\partial^3 F}{\partial y^3} = 0$$

generates a solution of the Boussinesq equation in the form

$$w = 12\frac{d}{dx}K(x,x;t),$$

where $K(x,y;t)$ is a solution of the linear Gel'fand–Levitan–Marchenko integral equation

$$K(x,y;t) + F(x,y;t) + \int_x^\infty K(x,s;t)F(s,y;t)\,ds = 0.$$

Time t appears here as a parameter.

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