24. $yy'_x = y + f(x)$.

Abel equation (Abel differential equation) of the second kind in the canonical form.

1. Solvable Abel equations. Tables 1–4 list all the Abel equations whose solutions are outlined in *Handbook of Exact Solutions for Ordinary Differential Equations* by Polyanin & Zaitsev. Tables 1–3 classify Abel equations in which the functions f are of the same form; Table 4 gives other Abel equations. In Table 1, equations are arranged in accordance with the growth of the parameter m. In Table 2, equations are arranged in accordance with the growth of the parameter p. In Table 3, equations are arranged in accordance with the growth of the parameter s. The rightmost column of the tables indicates the equation numbers where the corresponding solutions are written out.

TABLE 1
Solvable Abel equations of the form $yy'_x - y = sx + Ax^m$, A is an arbitrary parameter

m	s	Equation	m	s	Equation
any	$-\dfrac{2(m+1)}{(m+3)^2}$	1.3.1.10	-1	0	1.3.1.16
-7	$15/4$	1.3.1.56	$-1/2$	$-2/9$	1.3.1.26
-4	6	1.3.1.54	$-1/2$	$-4/25$	1.3.1.22
$-5/2$	12	1.3.1.47	$-1/2$	0	1.3.1.32
-2	0	1.3.1.33	$-1/2$	20	1.3.1.55
-2	2	1.3.1.19	0	any	1.3.1.2
$-5/3$	$-3/16$	1.3.1.30	0	0	1.3.1.1
$-5/3$	$-9/100$	1.3.1.23	$1/2$	$-12/49$	1.3.1.53
$-5/3$	$63/4$	1.3.1.48	2	$-6/25$	1.3.1.45
$-7/5$	$-5/36$	1.3.1.27	2	$6/25$	1.3.1.46

TABLE 2
Solvable Abel equations of the form $yy'_x - y = sx + \alpha Ax^p + \beta A^2 x^q$, A is an arbitrary parameter

p	q	s	α	β	Equation
-1	-3	any	1	-1	1.3.1.5
-1	-3	$\dfrac{2m+1}{4m^2}$	1	-1	1.3.1.13
-1	-3	0	1	-1	1.3.1.7
$-3/5$	$-7/5$	$-5/36$	any	any	1.3.1.62
$-5/11$	$-13/11$	$-33/196$	$286A/3$	$-770A/9$	1.3.1.69
$-1/3$	$-5/3$	$-3/16$	any	any	1.3.1.61
$-1/3$	$-5/3$	$-3/16$	3	-12	1.3.1.40
$-1/3$	$-5/3$	$-3/16$	5	-12	1.3.1.15
$-1/3$	$-5/3$	$15/4$	6	-3	1.3.1.60
$-1/5$	$-4/5$	$-10/49$	$13A/5$	$-7A/20$	1.3.1.68
0	$-1/2$	$-2/9$	any	any	1.3.1.3
2	3	$4/9$	2	2	1.3.1.14

TABLE 3
Solvable Abel equations of the form $yy'_x - y = sx + \sigma A(\alpha x^{1/2} + \beta A + \gamma A^2 x^{-1/2})$,
A is an arbitrary parameter

s	σ	α	β	γ	Equation
any $\neq 0$	any	0	any	0	1.3.1.2
$\frac{2(m-1)}{(m-3)^2}$	$\frac{2}{(m-3)^2}$	$m(m+3)$	$4m^2+3m+9$	$3m(m+3)$	1.3.1.12
$-1/4$	$1/4$	1	5	3	1.3.1.17
$-30/121$	$3/242$	21	35	6	1.3.1.29
$-12/49$	any	any	0	0	1.3.1.53
$-12/49$	$1/98$	25	41	10	1.3.1.25
$-12/49$	$6/49$	1	8	5	1.3.1.38
$-12/49$	$2/49$	5	34	15	1.3.1.24
$-12/49$	$4/49$	-10	27	10	1.3.1.31
$-12/49$	$1/49$	5	262	65	1.3.1.52
$-12/49$	$6/49$	-3	23	12	1.3.1.28
$-12/49$	$2/49$	1	166	55	1.3.1.58
$-12/49$	1	$3/49+3B$	$12/49-15B/2$	$15/196+75B/16$	1.3.1.64
$-6/25$	$2/25$	2	19	6	1.3.1.20
$-6/25$	$6/25$	2	7	4	1.3.1.39
$-28/121$	$2/121$	5	106	15	1.3.1.51
$-2/9$	any	0	any	any	1.3.1.3
$-2/9$	any	0	0	any	1.3.1.26
$-2/9$	6	0	1	2	1.3.1.11
$-10/49$	$2/49$	4	61	12	1.3.1.57
$-4/25$	any	0	0	any	1.3.1.22
$-4/25$	$1/50$	7	49	6	1.3.1.59
0	any	0	0	any	1.3.1.32
0	1	1	2	any	1.3.1.36
0	$n+2$	1	$2(n+2)$	$(n+1)(n+3)$	1.3.1.34
0	$n+2$	1	$2(n+2)$	$2n+3$	1.3.1.35
0	1	-1	2	0	1.3.1.37
0	2	1	4	3	1.3.1.4
0	any	0	any	0	1.3.1.1
2	2	-10	19	30	1.3.1.50
2	2	10	31	30	1.3.1.49
20	any	0	0	any	1.3.1.55

TABLE 4
Other solvable Abel equations of the form $yy'_x - y = f(x)$

Function $f(x)$	Equation
$Ax^{k-1} - kBx^k + kB^2x^{2k-1}$	1.3.1.6 (particular solution)
$Ax^2 - \frac{9}{625}A^{-1}$	1.3.1.44
$\frac{3}{4}x - \frac{3}{2}Ax^{1/3} + \frac{3}{4}A^2x^{-1/3} - \frac{27}{625}A^4x^{-5/3}$	1.3.1.66
$-\frac{6}{25}x + \frac{7}{5}Ax^{1/3} + \frac{31}{3}A^2x^{-1/3} - \frac{100}{3}A^4x^{-5/3}$	1.3.1.67
$-\frac{6}{25}x + ax^{1/3} + b + cx^{-1/3} + dx^{-2/3}$ (coefficients a, b, c, and d are related by an equality)	1.3.1.65
$-\frac{21}{100}x + \frac{7}{9}A^2\left(123x^{-1/7} + 280\,Ax^{-5/7} - 400\,A^2x^{-9/7}\right)$	1.3.1.70
$\dfrac{k}{\sqrt{Ax^2 + Bx + C}}$	1.3.1.63
$\dfrac{A}{\sqrt{x^2 + 4A}}$	1.3.1.18
$-\dfrac{3}{32}x + \dfrac{9a^2 - 6x^2}{64\sqrt{x^2 + a^2}}$	1.3.1.43
$\dfrac{3}{8}x + \dfrac{6x^2 + 5a^2}{16\sqrt{x^2 + a^2}}$	1.3.1.21
$\dfrac{3}{8}x + \dfrac{6x^2 + 9A}{16\sqrt{x^2 + A}}$	1.3.1.41
$\dfrac{9}{32}x + \dfrac{30x^2 + 33A}{64\sqrt{x^2 + A}}$	1.3.1.42
$A + B\exp(-2x/A)$	1.3.1.8
$A[\exp(2x/A) - 1]$	1.3.1.9
$a^2\lambda e^{2\lambda x} - a(b\lambda + 1)e^{\lambda x} + b$	1.3.1.73 (particular solution)
$a^2\lambda e^{2\lambda x} + a\lambda xe^{\lambda x} + be^{\lambda x}$	1.3.1.74 (particular solution)
$2a^2\lambda\sin(2\lambda x) + 2a\sin(\lambda x)$	1.3.1.75 (particular solution)

2. Use of particular solutions to construct the general solution. For some Abel equations of the second kind, the general solution can be found if n its distinct particular solutions $y_k = y_k(x)$, $k = 1, \dots, n$, are known.

Below we consider Abel equations of the canonical form

$$yy'_x - y = f(x), \tag{1}$$

whose general solutions can be represented in the special form:

$$\prod_{k=1}^{n} |y - y_k(x)|^{m_k} = C. \tag{2}$$

Here, the particular solutions $y_k = y_k(x)$ correspond to $C = 0$ (if $m_k > 0$) and $C = \infty$ (if $m_k < 0$).

Taking the logarithm of (2), followed by differentiating the resulting expression and rearrangement, leads to the equation

$$\sum_{j=1}^{n}\left[m_j(y'_x - y'_j)\prod_{\substack{k=1\\k\neq j}}^{n}(y-y_k)\right] \equiv y'_x\sum_{s=1}^{n-1}\Phi_s y^s + \sum_{s=1}^{n-1}\Psi_s y^s = 0, \tag{3}$$

where $y'_j = (y_j)'_x$. We require that equation (3) be equivalent to the Abel equation (1). To this end, we set:

$$\Psi_\nu = -\Phi_\nu, \quad \Psi_{\nu-1} = -f(x)\Phi_\nu \quad \text{and equate the other } \Phi_i \text{ and } \Psi_i \text{ with zero.}$$

Selecting different values $\nu = 1,\ 2,\ \ldots,\ n-1$, we obtain $n-1$ systems of differential-algebraic equations; only one of the systems, corresponding to $m_k \neq 0$ for all $k = 1,\ldots,n$ and $y_i \neq y_j$ for $i \neq j$, leads to a nondegenerate solution of the form (2). Consider the Abel equations (1) corresponding to the simplest solutions of the form (2) in more detail.

1°. *Case* $n = 2$. The system of differential-algebraic equations has the form:

$$\begin{aligned} &m_1 + m_2 = M,\\ &m_1y_2 + m_2y_1 = 0,\\ &m_1y'_1 + m_2y'_2 = M,\\ &m_1y'_1y_2 + m_2y_1y'_2 = -Mf(x), \end{aligned} \tag{4}$$

where M is an arbitrary constant. It follows from the second and third equations that

$$y_1 = \frac{m_1}{m_1^2 - m_2^2}(Mx+N), \quad y_2 = -\frac{m_2}{m_1^2 - m_2^2}(Mx+N),$$

where N is an arbitrary constant. Introducing the new constants

$$A = \frac{m_1m_2(m_1+m_2)}{(m_1-m_2)^2}M, \quad B = \frac{m_1m_2(m_1+m_2)}{(m_1-m_2)^2}N,$$

we find from the last relation in (4) that

$$f(x) = Ax + B. \tag{5}$$

The particular solutions y_1, y_2, and the corresponding exponents m_1, m_2 in the general integral (2), are expressed in terms of the coefficients A, B on the right-hand side (5) of the Abel equation (1) as follows:

$$\begin{aligned} y_1 &= \frac{1+\sqrt{4A+1}}{2A}(Ax+B), && m_1 = 2A + 1 + \sqrt{4A+1},\\ y_2 &= -\frac{1+\sqrt{4A+1}}{2A+1+\sqrt{4A+1}}(Ax+B), && m_2 = 2A. \end{aligned}$$

2°. *Case* $n = 3$. Equation (3) with $n = 3$ leads to the Abel equation (1) with the right-hand side

$$f(x) = -\tfrac{2}{9}x + A + Bx^{-1/2}. \tag{6}$$

The particular solutions and the exponents in the general integral (2) are expressed as:

$$y_s = \frac{2}{3}x + \frac{2}{3}\lambda_s x^{1/2} + \frac{3B}{\lambda_s}, \quad m_s = \frac{2A}{3(2\lambda_s^2 - 3A)},$$

where the λ_s $(s = 1,\ 2,\ 3)$ are roots of the cubic equation

$$\lambda^3 - \tfrac{9}{2}A\lambda - \tfrac{9}{2}B = 0.$$

3°. *Case* $n = 4$. Equation (3) with $n = 4$ leads to the Abel equation (1) with the right-hand side

$$f(x) = -\tfrac{3}{16}x + Ax^{-1/3} + Bx^{-5/3}.$$

The particular solutions and the exponents in (2) are expressed as:

$$y_{1,2} = \tfrac{3}{4}x \pm \sqrt{3A + \tfrac{3}{2}\sqrt{-3B}}\,x^{1/3} + \sqrt{-3B}\,x^{-1/3}, \quad m_{1,2} = \mp\bigl(2A - \sqrt{-3B}\,\bigr),$$
$$y_{3,4} = \tfrac{3}{4}x \pm \sqrt{3A - \tfrac{3}{2}\sqrt{-3B}}\,x^{1/3} - \sqrt{-3B}\,x^{-1/3}, \quad m_{3,4} = \pm\sqrt{4A^2 + 3B}.$$

3. Other Abel equations. See also:

- Abel differential equation of the second kind (special case) ,
- Abel differential equation of the second kind .

References

Zaitsev, V. F. and Polyanin, A. D., *Discrete-Group Methods for Integrating Equations of Nonlinear Mechanics* , CRC Press, Boca Raton, 1994.

Polyanin, A. D. and Zaitsev, V. F., *Handbook of Exact Solutions for Ordinary Differential Equations, 2nd Edition* , Chapman & Hall/CRC, Boca Raton, 2003.