25. $yy'_x = f(x)y + g(x)$**.**

Abel differential equation of the second kind (special case).

1°. The substitution $z = \int f(x)\,dx$ brings the Abel equation to the canonical form:

$$yy'_z = y + \Phi(z), \qquad (1)$$

where the function $\Phi(z)$ is defined parametrically (x is the parameter) by the relations

$$\Phi = \frac{g(x)}{f(x)}, \qquad z = \int f(x)\,dx.$$

Solvable Abel equations of the form (1) see here.

2°. The substitution $\xi = \int g(x)\,dx$ brings the Abel equation to the simpler form:

$$yy'_\xi = \Psi(\xi)y + 1, \qquad (2)$$

where the function $\Psi(\xi)$ is defined parametrically by the relations

$$\Psi = \frac{f(x)}{g(x)}, \qquad \xi = \int g(x)\,dx.$$

3°. The books by Zaitsev & Polyanin (1994) and Polyanin & Zaitsev (2003) present a large number of solutions to the Abel equation of the second kind for various $f(x)$ and $g(x)$.

Example. Consider the Abel equation

$$yy'_x = [a(2n+k)x^k + b]x^{n-1}y + (-a^2nx^{2k} - abx^k + c)x^{2n-1}.$$

The substitution $y = x^n(w + ax^k)$ leads to a Bernoulli equation with respect to $x = x(w)$:

$$(nw^2 - bw - c)x'_w = -wx - ax^{k+1}.$$

References

Zaitsev, V. F. and Polyanin, A. D., *Discrete-Group Methods for Integrating Equations of Nonlinear Mechanics*, CRC Press, Boca Raton, 1994.

Polyanin, A. D. and Zaitsev, V. F., *Handbook of Exact Solutions for Ordinary Differential Equations, 2nd Edition*, Chapman & Hall/CRC, Boca Raton, 2003.

