26. $yy'_x = f(x)y^2 + g(x)y + h(x).$

Abel differential equation of the second kind.

1°. The substitution

$$y = E(x)w, \quad \text{where} \quad E(x) = \exp\Bigl(\int f(x)\,dx\Bigr),$$

brings this equation to the simpler form:

$$ww'_x = F_1(x)w + F_0(x), \tag{1}$$

where

$$F_1(x) = g(x)/E(x), \quad F_0(x) = h(x)/E^2(x).$$

2°. In turn, equation (1) can be reduced, by the introduction of the new independent variable

$$z = \int F_1(x)\,dx,$$

to the canonical form:

$$ww'_z - w = \Phi(z). \tag{2}$$

Here, the function $\Phi(z)$ is defined parametrically (x is the parameter) by the relations

$$\Phi = \frac{F_0(x)}{F_1(x)}, \quad z = \int F_1(x)\,dx.$$

Remark. The transformation $w = a\hat{w}$, $z = a\hat{z} + b$ brings (2) to a similar equation, $\hat{w}\hat{w}'_{\hat{z}} - \hat{w} = a^{-1}\Phi(a\hat{z} + b)$. Therefore the function $\Phi(z)$ in the right-hand side of the Abel equation (2) can be identified with the two-parameter family of functions $a^{-1}\Phi(az + b)$.

Remark 2. The books by Zaitsev & Polyanin (1994) and Polyanin & Zaitsev (2003) present a large number of solutions to the Abel equations of the forms (1) and (2). Solvable Abel equations of the form (2) see here .

References

Kamke, E., *Differentialgleichungen: Lösungsmethoden und Lösungen, I, Gewöhnliche Differentialgleichungen*, B. G. Teubner, Leipzig, 1977.

Zaitsev, V. F. and Polyanin, A. D., *Discrete-Group Methods for Integrating Equations of Nonlinear Mechanics* , CRC Press, Boca Raton, 1994.

Polyanin, A. D. and Zaitsev, V. F., *Handbook of Exact Solutions for Ordinary Differential Equations, 2nd Edition* , Chapman & Hall/CRC, Boca Raton, 2003.

