Abel differential equation of the second kind.

1°. The substitution

\[ y = E(x)w, \quad \text{where} \quad E(x) = \exp\left(\int f(x) \, dx\right), \]

brings this equation to the simpler form:

\[ w'w = F_1(x)w + F_0(x), \quad (1) \]

where

\[ F_1(x) = g(x)/E(x), \quad F_0(x) = h(x)/E^2(x). \]

2°. In turn, equation (1) can be reduced, by the introduction of the new independent variable

\[ z = \int F_1(x) \, dx, \]

to the canonical form:

\[ w'w - w = \Phi(z). \quad (2) \]

Here, the function \( \Phi(z) \) is defined parametrically (\( x \) is the parameter) by the relations

\[ \Phi = \frac{F_0(x)}{F_1(x)}, \quad z = \int F_1(x) \, dx. \]

Remark. The transformation \( w = a\hat{w} \), \( z = a\hat{z} + b \) brings (2) to a similar equation, \( \hat{w}'\hat{w} - \hat{w} = a^{-1}\Phi(a\hat{z} + b) \). Therefore the function \( \Phi(z) \) in the right-hand side of the Abel equation (2) can be identified with the two-parameter family of functions \( a^{-1}\Phi(az + b) \).

Remark 2. The books by Zaitsev & Polyanin (1994) and Polyanin & Zaitsev (2003) present a large number of solutions to the Abel equations of the forms (1) and (2). Solvable Abel equations of the form (2) [see here].

References

