31. $y'_x = f\left(\dfrac{ax + by + c}{\alpha x + \beta y + \gamma}\right).$

1°. For $\Delta = a\beta - b\alpha \neq 0$, the transformation

$$x = u + \frac{b\gamma - c\beta}{\Delta}, \qquad y = v(u) + \frac{c\alpha - a\gamma}{\Delta}$$

leads to an equation:

$$v'_u = f\left(\frac{au + bv}{\alpha u + \beta v}\right).$$

Dividing both the numerator and denominator of the fraction on the right-hand side by u, we obtain a homogeneous equation of the form 1.5.

2°. For $\Delta = 0$ and $b \neq 0$, the substitution $v(x) = ax + by + c$ leads to a separable equation:

$$v'_x = a + bf\left(\frac{bv}{\beta v + b\gamma - c\beta}\right).$$

3°. For $\Delta = 0$ and $\beta \neq 0$, the substitution $v(x) = \alpha x + \beta y + \gamma$ also leads to a separable equation:

$$v'_x = \alpha + \beta f\left(\frac{bv + c\beta - b\gamma}{\beta v}\right).$$

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