13. $x^2 y''_{xx} + x y'_x + (x^2 - \nu^2) y = 0.$

Bessel equation.

1°. Let ν be an arbitrary noninteger. Then the general solution is given by:

$$y = C_1 J_\nu(x) + C_2 Y_\nu(x), \tag{1}$$

where C_1 and C_2 are arbitrary constants, $J_\nu(x)$ and $Y_\nu(x)$ are the Bessel functions of the first and second kind:

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)}, \qquad Y_\nu(x) = \frac{J_\nu(x)\cos\pi\nu - J_{-\nu}(x)}{\sin\pi\nu}. \tag{2}$$

Solution (1) is denoted by $y = Z_\nu(x)$ which is referred to as the cylindrical function.

The functions $J_\nu(x)$ and $Y_\nu(x)$ can be expressed in terms of definite integrals (with $x > 0$):

$$\pi J_\nu(x) = \int_0^\pi \cos(x\sin\theta - \nu\theta)\,d\theta - \sin\pi\nu \int_0^\infty \exp(-x\sinh t - \nu t)\,dt,$$

$$\pi Y_\nu(x) = \int_0^\pi \sin(x\sin\theta - \nu\theta)\,d\theta - \int_0^\infty (e^{\nu t} + e^{-\nu t}\cos\pi\nu) e^{-x\sinh t}\,dt.$$

2°. In the case $\nu = n + \frac{1}{2}$, where $n = 0, 1, 2, \ldots,$ the Bessel functions are expressed in terms of elementary functions:

$$J_{n+\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}}\, x^{n+\frac{1}{2}} \left(-\frac{1}{x}\frac{d}{dx}\right)^n \frac{\sin x}{x}, \qquad J_{-n-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}}\, x^{n+\frac{1}{2}} \left(\frac{1}{x}\frac{d}{dx}\right)^n \frac{\cos x}{x},$$

$$Y_{n+\frac{1}{2}}(x) = (-1)^{n+1} J_{-n-\frac{1}{2}}(x).$$

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