22. $x(x-1)y''_{xx} + [(\alpha+\beta+1)x-\gamma]y'_x + \alpha\beta y = 0.$

Gaussian hypergeometric equation. For $\gamma \neq 0, -1, -2, -3, \ldots$, a solution can be expressed in terms of the hypergeometric series:

$$F(\alpha,\beta,\gamma;x) = 1 + \sum_{k=1}^{\infty} \frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{x^k}{k!}, \qquad (\alpha)_k = \alpha(\alpha+1)\ldots(\alpha+k-1),$$

which, *a fortiori*, is convergent for $|x| < 1$.

For $\gamma > \beta > 0$, this solution can be expressed in terms of a definite integral:

$$F(\alpha,\beta,\gamma;x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tx)^{-\alpha}\,dt,$$

where $\Gamma(\beta)$ is the gamma function.

If γ is not an integer, the general solution of the hypergeometric equation has the form:

$$y = C_1F(\alpha,\beta,\gamma;x) + C_2x^{1-\gamma}F(\alpha-\gamma+1,\ \beta-\gamma+1,\ 2-\gamma;\ x).$$

Table gives the general solutions of the hypergeometric equation for some values of the determining parameters.

TABLE

General solutions of the hypergeometric equation for some values of the determining parameters

α	β	γ	Solution: $y=y(x)$
0	β	γ	$C_1+C_2\int \lvert x\rvert^{-\gamma}\lvert x-1\rvert^{\gamma-\beta-1}\,dx$
α	$\alpha+\frac{1}{2}$	$2\alpha+1$	$C_1\bigl(1+\sqrt{1-x}\bigr)^{-2\alpha}+C_2x^{-2\alpha}\bigl(1+\sqrt{1-x}\bigr)^{2\alpha}$
α	$\alpha-\frac{1}{2}$	$\frac{1}{2}$	$C_1\bigl(1+\sqrt{x}\bigr)^{1-2\alpha}+C_2\bigl(1-\sqrt{x}\bigr)^{1-2\alpha}$
α	$\alpha+\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{\sqrt{x}}\Bigl[C_1\bigl(1+\sqrt{x}\bigr)^{1-2\alpha}+C_2\bigl(1-\sqrt{x}\bigr)^{1-2\alpha}\Bigr]$
1	β	γ	$\lvert x\rvert^{1-\gamma}\lvert x-1\rvert^{\gamma-\beta-1}\Bigl(C_1+C_2\int \lvert x\rvert^{\gamma-2}\lvert x-1\rvert^{\beta-\gamma}\,dx\Bigr)$
α	β	α	$\lvert x-1\rvert^{-\beta}\Bigl(C_1+C_2\int \lvert x\rvert^{-\alpha}\lvert x-1\rvert^{\beta-1}\,dx\Bigr)$
α	β	$\alpha+1$	$\lvert x\rvert^{-\alpha}\Bigl(C_1+C_2\int \lvert x\rvert^{\alpha-1}\lvert x-1\rvert^{-\beta}\,dx\Bigr)$

References

Bateman, H. and Erdélyi, A., *Higher Transcendental Functions*, Vol. 1, McGraw-Hill, New York, 1953.

Abramowitz, M. and Stegun, I. A. (Editors), *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, National Bureau of Standards Applied Mathematics, Washington, 1964.

Polyanin, A. D. and Zaitsev, V. F., *Handbook of Exact Solutions for Ordinary Differential Equations, 2nd Edition* , Chapman & Hall/CRC, Boca Raton, 2003.

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