34. $y''_{xx} + (a - 2q\cos 2x)y = 0.$

Mathieu equation.

1°. Given numbers a and q, there exists a general solution $y(x)$ and a characteristic index μ such that

$$y(x+\pi) = e^{2\pi\mu} y(x).$$

For small values of q, an approximate value of μ can be found from the equation:

$$\cosh(\pi\mu) = 1 + 2\sin^2\left(\tfrac{1}{2}\pi\sqrt{a}\,\right) + \frac{\pi q^2}{(1-a)\sqrt{a}}\sin\left(\pi\sqrt{a}\,\right) + O(q^4).$$

If $y_1(x)$ is the solution of the Mathieu equation satisfying the initial conditions $y_1(0)=1$ and $y_1'(0)=0$, the characteristic index can be determined from the relation:

$$\cosh(2\pi\mu) = y_1(\pi).$$

The solution $y_1(x)$, and hence μ, can be determined with any degree of accuracy by means of numerical or approximate methods.

The general solution differs depending on the value of $y_1(\pi)$ and can be expressed in terms of two auxiliary periodical functions $\varphi_1(x)$ and $\varphi_2(x)$ (see Table 1).

TABLE 1
The general solution of the Mathieu equation expressed
in terms of auxiliary periodical functions $\varphi_1(x)$ and $\varphi_2(x)$

Constraint	General solution $y = y(x)$	Period of φ_1 and φ_2	Index
$y_1(\pi) > 1$	$C_1 e^{2\mu x}\varphi_1(x) + C_2 e^{-2\mu x}\varphi_2(x)$	π	μ is a real number
$y_1(\pi) < -1$	$C_1 e^{2\rho x}\varphi_1(x) + C_2 e^{-2\rho x}\varphi_2(x)$	2π	$\mu = \rho + \frac{1}{2}i$, $i^2 = -1$, ρ is the real part of μ
$\lvert y_1(\pi)\rvert < 1$	$(C_1\cos\nu x + C_2\sin\nu x)\varphi_1(x) + (C_1\cos\nu x - C_2\sin\nu x)\varphi_2(x)$	π	$\mu = i\nu$ is a pure imaginary number, $\cos(2\pi\nu) = y_1(\pi)$
$y_1(\pi) = \pm 1$	$C_1\varphi_1(x) + C_2 x\varphi_2(x)$	π	$\mu = 0$

2°. In applications, of major interest are periodical solutions of the Mathieu equation that exist for certain values of the parameters a and q (those values of a are referred to as eigenvalues). The most important solutions are listed in Table 2.

The Mathieu functions possess the following properties:

$$\mathrm{ce}_{2n}(x, -q) = (-1)^n\,\mathrm{ce}_{2n}\left(\frac{\pi}{2} - x,\ q\right), \qquad \mathrm{ce}_{2n+1}(x, -q) = (-1)^n\,\mathrm{se}_{2n+1}\left(\frac{\pi}{2} - x,\ q\right),$$
$$\mathrm{se}_{2n}(x, -q) = (-1)^{n-1}\,\mathrm{se}_{2n}\left(\frac{\pi}{2} - x,\ q\right), \qquad \mathrm{se}_{2n+1}(x, -q) = (-1)^n\,\mathrm{ce}_{2n+1}\left(\frac{\pi}{2} - x,\ q\right).$$

Selecting a sufficiently large m and omitting the term with the maximum number in the recurrence relations (indicated in Table 20), we can obtain approximate relations for the eigenvalues a_n (or b_n)

TABLE 2

Periodical solutions of the Mathieu equation $\mathrm{ce}_n = \mathrm{ce}_n(x,q)$ and $\mathrm{se}_n = \mathrm{se}_n(x,q)$ (for odd n, the functions ce_n and se_n are 2π-periodical, and for even n, they are π-periodical); certain eigenvalues $a = a_n(q)$ and $b = b_n(q)$ correspond to each value of the parameter q; $n = 0,\ 1,\ 2,\ \ldots$

Mathieu functions	Recurrence relations for coefficients	Normalization conditions
$\mathrm{ce}_{2n}(x,q)=\sum\limits_{m=0}^{\infty} A_{2m}^{2n}\cos(2mx)$	$qA_2^{2n}=a_{2n}A_0^{2n}$; $qA_4^{2n}=(a_{2n}-4)A_2^{2n}-2qA_0^{2n}$; $qA_{2m+2}^{2n}=(a_{2n}-4m^2)A_{2m}^{2n} - qA_{2m-2}^{2n}$, $m\ge 2$	$(A_0^{2n})^2+\sum\limits_{m=0}^{\infty}\bigl(A_{2m}^{2n}\bigr)^2 =\begin{cases} 2 & \text{if } n=0 \\ 1 & \text{if } n\ge 1\end{cases}$
$\mathrm{ce}_{2n+1}(x,q)=\sum\limits_{m=0}^{\infty} A_{2m+1}^{2n+1}\cos\bigl[(2m+1)x\bigr]$	$qA_3^{2n+1}=(a_{2n+1}-1-q)A_1^{2n+1}$; $qA_{2m+3}^{2n+1}=[a_{2n+1}-(2m+1)^2] \times A_{2m+1}^{2n+1}-qA_{2m-1}^{2n+1}$, $m\ge 1$	$\sum\limits_{m=0}^{\infty}\bigl(A_{2m+1}^{2n+1}\bigr)^2=1$
$\mathrm{se}_{2n}(x,q)=\sum\limits_{m=0}^{\infty} B_{2m}^{2n}\sin(2mx)$, $\mathrm{se}_0=0$	$qB_4^{2n}=(b_{2n}-4)B_2^{2n}$; $qB_{2m+2}^{2n}=(b_{2n}-4m^2)B_{2m}^{2n} - qB_{2m-2}^{2n}$, $m\ge 2$	$\sum\limits_{m=0}^{\infty}\bigl(B_{2m}^{2n}\bigr)^2=1$
$\mathrm{se}_{2n+1}(x,q)=\sum\limits_{m=0}^{\infty} B_{2m+1}^{2n+1}\sin\bigl[(2m+1)x\bigr]$	$qB_3^{2n+1}=(b_{2n+1}-1-q)B_1^{2n+1}$; $qB_{2m+3}^{2n+1}=[b_{2n+1}-(2m+1)^2] \times B_{2m+1}^{2n+1}-qB_{2m-1}^{2n+1}$, $m\ge 1$	$\sum\limits_{m=0}^{\infty}\bigl(B_{2m+1}^{2n+1}\bigr)^2=1$

with respect to parameter q. Then, equating the determinant of the corresponding homogeneous linear system of equations for coefficients A_m^n (or B_m^n) to zero, we obtain an algebraic equation for finding $a_n(q)$ (or $b_n(q)$).

For fixed real $q \neq 0$, the eigenvalues a_n and b_n are all real and different, while:

$$\begin{aligned} &\text{if}\quad q>0\quad \text{then}\quad a_0<b_1<a_1<b_2<a_2<\cdots; \\ &\text{if}\quad q<0\quad \text{then}\quad a_0<a_1<b_1<b_2<a_2<a_3<b_3<b_4<\cdots \end{aligned}$$

The eigenvalues possess the following properties:

$$a_{2n}(-q)=a_{2n}(q),\qquad b_{2n}(-q)=b_{2n}(q),\qquad a_{2n+1}(-q)=b_{2n+1}(q).$$

The solution of the Mathieu equation corresponding to eigenvalue a_n (or b_n) has n zeros on the interval $0 \le x < \pi$ (q is a real number).

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