34. \( y'' + (a - 2q \cos 2x)y = 0 \).

**Mathieu equation.**

1°. Given numbers \( a \) and \( q \), there exists a general solution \( y(x) \) and a characteristic index \( \mu \) such that

\[
y(x + \pi) = e^{2\pi \mu} y(x).
\]

For small values of \( q \), an approximate value of \( \mu \) can be found from the equation:

\[
\cosh(\pi \mu) = 1 + 2 \sin^2 \left( \frac{1}{2} \pi \sqrt{a} \right) + \frac{\pi q^2}{(1 - a)\sqrt{a}} \sin(\pi \sqrt{a}) + O(q^4).
\]

If \( y_1(x) \) is the solution of the Mathieu equation satisfying the initial conditions \( y_1(0) = 1 \) and \( y_1'(0) = 0 \), the characteristic index can be determined from the relation:

\[
\cosh(2\pi \mu) = y_1(\pi).
\]

The solution \( y_1(x) \), and hence \( \mu \), can be determined with any degree of accuracy by means of numerical or approximate methods.

The general solution differs depending on the value of \( y_1(\pi) \) and can be expressed in terms of two auxiliary periodical functions \( \varphi_1(x) \) and \( \varphi_2(x) \) (see Table 1).

<table>
<thead>
<tr>
<th>Constraint</th>
<th>General solution ( y = y(x) )</th>
<th>Period of ( \varphi_1 ) and ( \varphi_2 )</th>
<th>Index</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y_1(\pi) &gt; 1 )</td>
<td>( C_1 e^{2\mu x} \varphi_1(x) + C_2 e^{-2\mu x} \varphi_2(x) )</td>
<td>( \pi )</td>
<td>( \mu ) is a real number</td>
</tr>
<tr>
<td>( y_1(\pi) &lt; -1 )</td>
<td>( C_1 e^{2\mu x} \varphi_1(x) + C_2 e^{-2\mu x} \varphi_2(x) )</td>
<td>( 2\pi )</td>
<td>( \mu = \rho + \frac{1}{2}i, ; i^2 = -1, ; \rho ) is the real part of ( \mu )</td>
</tr>
<tr>
<td>(</td>
<td>y_1(\pi)</td>
<td>&lt; 1 )</td>
<td>( (C_1 \cos \nu x + C_2 \sin \nu x)\varphi_1(x) + (+C_1 \cos \nu x - C_2 \sin \nu x)\varphi_2(x) )</td>
</tr>
<tr>
<td>( y_1(\pi) = \pm 1 )</td>
<td>( C_1 \varphi_1(x) + C_2 x \varphi_2(x) )</td>
<td>( \pi )</td>
<td>( \mu = 0 )</td>
</tr>
</tbody>
</table>

2°. In applications, of major interest are periodical solutions of the Mathieu equation that exist for certain values of the parameters \( a \) and \( q \) (those values of \( a \) are referred to as eigenvalues). The most important solutions are listed in Table 2.

The Mathieu functions possess the following properties:

\[
\begin{align*}
\text{ce}_{2n}(x, -q) &= (-1)^n \text{ce}_{2n}\left( \frac{\pi}{2} - x, q \right), \\
\text{se}_{2n}(x, -q) &= (-1)^{n-1} \text{se}_{2n}\left( \frac{\pi}{2} - x, q \right),
\end{align*}
\]

\[
\begin{align*}
\text{ce}_{2n+1}(x, -q) &= (-1)^n \text{se}_{2n+1}\left( \frac{\pi}{2} - x, q \right),
\end{align*}
\]

Selecting a sufficiently large \( m \) and omitting the term with the maximum number in the recurrence relations (indicated in Table 20), we can obtain approximate relations for the eigenvalues \( a_n \) (or \( b_n \))
Periodical solutions of the Mathieu equation $ce_n = ce_n(x, q)$ and $se_n = se_n(x, q)$ (for odd $n$, the functions $ce_n$ and $se_n$ are $2\pi$-periodical, and for even $n$, they are $\pi$-periodical); certain eigenvalues $a = a_n(q)$ and $b = b_n(q)$ correspond to each value of the parameter $q$; $n = 0, 1, 2, \ldots$

<table>
<thead>
<tr>
<th>Mathieu functions</th>
<th>Recurrence relations for coefficients</th>
<th>Normalization conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ce_{2n}(x, q) = \sum_{m=0}^{\infty} A_{2m}^{2n} \cos(2mx)$</td>
<td>$qA_{2n}^{2n} = a_{2n} A_{0}^{2n}$; $qA_{4}^{2n} = (a_{2n} - 4) A_{2}^{2n} - 2q A_{0}^{2n}$; $qA_{2m}^{2n} = (a_{2n} - 4m^2) A_{2m}^{2n}$; $-qA_{2m-2}^{2n}$, $m \geq 2$</td>
<td>$(A_{0}^{2n})^2 + \sum_{m=0}^{\infty} (A_{2m}^{2n})^2 = 1$ if $n = 0$</td>
</tr>
<tr>
<td>$ce_{2n+1}(x, q) = \sum_{m=0}^{\infty} A_{2m+1}^{2n+1} \cos[(2m+1)x]$</td>
<td>$qA_{2n+1}^{2n+1} = (a_{2n+1} - 1 - q) A_{1}^{2n+1}$; $qA_{2m+3}^{2n+1} = [a_{2n+1} - (2m+1)^2]$ $\times A_{2m+1}^{2n+1}$ $- qA_{2m+1}^{2n+1}$, $m \geq 1$</td>
<td>$\sum_{m=0}^{\infty} (A_{2m+1}^{2n+1})^2 = 1$</td>
</tr>
<tr>
<td>$se_{2n}(x, q) = \sum_{m=0}^{\infty} B_{2m}^{2n} \sin(2mx)$, $se_0 = 0$</td>
<td>$qB_{2}^{2n} = (b_{2n} - 4) B_{2}^{2n}$; $qB_{2m}^{2n} = (b_{2n} - 4m^2) B_{2m}^{2n}$; $-qB_{2m-2}^{2n}$, $m \geq 2$</td>
<td>$\sum_{m=0}^{\infty} (B_{2m}^{2n})^2 = 1$</td>
</tr>
<tr>
<td>$se_{2n+1}(x, q) = \sum_{m=0}^{\infty} B_{2m+1}^{2n+1} \sin[(2m+1)x]$</td>
<td>$qB_{2n+1}^{2n+1} = (b_{2n+1} - 1 - q) B_{1}^{2n+1}$; $qB_{2m+3}^{2n+1} = [b_{2n+1} - (2m+1)^2]$ $\times B_{2m+1}^{2n+1}$ $- qB_{2m+1}^{2n+1}$, $m \geq 1$</td>
<td>$\sum_{m=0}^{\infty} (B_{2m+1}^{2n+1})^2 = 1$</td>
</tr>
</tbody>
</table>

with respect to parameter $q$. Then, equating the determinant of the corresponding homogeneous linear system of equations for coefficients $A_{2n}^{n}$ (or $B_{2n}^{n}$) to zero, we obtain an algebraic equation for finding $a_n(q)$ (or $b_n(q)$).

For fixed real $q \neq 0$, the eigenvalues $a_n$ and $b_n$ are all real and different, while:

- if $q > 0$, then $a_0 < b_1 < a_1 < b_2 < a_2 < \ldots$;
- if $q < 0$, then $a_0 < a_1 < b_1 < b_2 < a_2 < a_3 < b_3 < b_4 < \ldots$

The eigenvalues possess the following properties:

$$a_{2n}(-q) = a_{2n}(q), \quad b_{2n}(-q) = b_{2n}(q), \quad a_{2n+1}(-q) = b_{2n+1}(q).$$

The solution of the Mathieu equation corresponding to eigenvalue $a_n$ (or $b_n$) has $n$ zeros on the interval $0 \leq x < \pi$ ($q$ is a real number).

References


Mathieu Equation