12. $y_x^{(n)} = ax^\beta y.$

1°. The transformation $x = t^{-1}$, $y = wt^{1-n}$ leads to an equation of similar form:

$$w_t^{(n)} = a(-1)^{n+1} t^{-2n-\beta} w.$$

2°. Let $n \ge 2$, $\beta > -n$, and $(n+\beta)(s+1) \ne 1, 2, \ldots, n-1$, where $s = 0, 1, \ldots$ Then the equation has n solutions that can be represented as:

$$y_j(x) = x^{j-1} E_{n,1+\beta/n,(\beta+j-1)/n}\bigl(ax^{\beta+n}\bigr), \qquad j = 1, 2, \ldots, n. \tag{1}$$

Here, $E_{n,m,l}(z)$ is a Mittag-Leffler type special function defined by:

$$E_{n,m,l}(z) = 1 + \sum_{k=1}^{\infty} b_k z^k,$$
$$b_k = \prod_{s=0}^{k-1} \frac{\Gamma\bigl(n(ms+l)+1\bigr)}{\Gamma\bigl(n(ms+l+1)+1\bigr)} = \prod_{s=0}^{k-1} \frac{1}{[n(ms+l)+1]\ldots[n(ms+l)+n]}, \tag{2}$$

where $\Gamma(\xi)$ is the gamma function, l is an arbitrary number, and $m > 0$.

If $\beta \ge 0$, solutions (1) are linearly independent. Series expansions of (1) are convenient for small x.

3°. Let $n \ge 2$, $\beta < -n$, and $(n+\beta)(s+1) \ne -1, -2, \ldots, -(n-1)$, where $s = 0, 1, \ldots$ Then the equation in question has n solutions that can be represented as:

$$y_j(x) = x^{j-1} E_{n,-1-\beta/n,-1-(\beta+j)/n}\bigl(a(-1)^n x^{\beta+n}\bigr), \qquad j = 1, 2, \ldots, n, \tag{3}$$

where $E_{n,m,l}(z)$ is the Mittag-Leffler type special function defined by (2). If $\beta \le -2n$, solutions (3) are linearly independent. Series expansions of (3) are convenient for large x.

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