15. $a_n y_x^{(n)} + a_{n-1} y_x^{(n-1)} + \cdots + a_1 y_x' + a_0 y = 0, \qquad a_n \neq 0.$

Constant coefficient linear homogeneous differential equation. The general solution of this equation is determined by the roots of the characteristic equation:

$$P(\lambda) = 0, \qquad \text{where} \quad P(\lambda) = a_n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1 \lambda + a_0.$$

The following cases are possible:

1°. All roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the characteristic equation are real and distinct. Then the general solution of the homogeneous linear differential equation (1) has the form:

$$y = C_1 \exp(\lambda_1 x) + C_2 \exp(\lambda_2 x) + \cdots + C_n \exp(\lambda_n x).$$

2°. There are m equal real roots $\lambda_1 = \lambda_2 = \cdots = \lambda_m$ $(m \le n)$, and the other roots are real and distinct. In this case, the general solution is given by:

$$\begin{aligned} y &= \exp(\lambda_1 x)(C_1 + C_2 x + \cdots + C_m x^{m-1}) \\ &\quad + C_{m+1} \exp(\lambda_{m+1} x) + C_{m+2} \exp(\lambda_{m+2} x) + \cdots + C_n \exp(\lambda_n x). \end{aligned}$$

3°. There are m equal complex conjugate roots $\lambda = \alpha \pm i\beta$ $(2m \le n)$, and the other roots are real and distinct. In this case, the general solution is:

$$\begin{aligned} y &= \exp(\alpha x)\cos(\beta x)(A_1 + A_2 x + \cdots + A_m x^{m-1}) \\ &\quad + \exp(\alpha x)\sin(\beta x)(B_1 + B_2 x + \cdots + B_m x^{m-1}) \\ &\quad + C_{2m+1} \exp(\lambda_{2m+1} x) + C_{2m+2} \exp(\lambda_{2m+2} x) + \cdots + C_n \exp(\lambda_n x), \end{aligned}$$

where $A_1, \ldots, A_m, B_1, \ldots, B_m, C_{2m+1}, \ldots, C_n$ are arbitrary constants.

4°. In the general case, where there are r different roots $\lambda_1, \lambda_2, \ldots, \lambda_r$ of multiplicities $m_1, m_2, \ldots, m_r$, respectively, the right-hand side of the characteristic equation can be represented as the product

$$P(\lambda) = (\lambda - \lambda_1)^{m_1} (\lambda - \lambda_2)^{m_2} \ldots (\lambda - \lambda_r)^{m_r},$$

where $m_1 + m_2 + \cdots + m_r = n$. The general solution of the original equation is given by the formula:

$$y = \sum_{k=1}^{r} \exp(\lambda_k x)(C_{k,0} + C_{k,1} x + \cdots + C_{k,m_k-1} x^{m_k-1}),$$

where $C_{k,l}$ are arbitrary constants.

If the characteristic equation has complex conjugate roots, then in the above solution, one should extract the real part on the basis of the relation $\exp(\alpha \pm i\beta) = e^{\alpha}(\cos\beta \pm i \sin\beta)$.

References

Kamke, E., *Differentialgleichungen: Lösungsmethoden und Lösungen, I, Gewöhnliche Differentialgleichungen*, B. G. Teubner, Leipzig, 1977.

Boyce, W. E. and DiPrima, R. C., *Elementary Differential Equations, 7th Edition*, Wiley, New York, 2000.

Polyanin, A. D. and Zaitsev, V. F., *Handbook of Exact Solutions for Ordinary Differential Equations, 2nd Edition* , Chapman & Hall/CRC, Boca Raton, 2003.

