1. \( x'_1 = ax + by, \quad y'_1 = cx + dy. \)

*System of two linear homogeneous first-order constant-coefficient ordinary differential equations.*

The characteristic equation is written as

\[
\lambda^2 - (a + d)\lambda + ad - bc = 0
\]

and its discriminant is

\[
D = (a - d)^2 + 4bc.
\]

Consider several cases.

1\(^o\). Case \( ad - bc \neq 0 \). The origin of coordinates, \( x = y = 0 \), is the only stationary point; it is:

- a node if \( D = 0 \);
- a saddle if \( D > 0, \quad ad - bc < 0 \);
- a focus if \( D < 0, \quad a + d \neq 0 \);
- a center if \( D < 0, \quad a + d = 0 \).

1.1. Let \( D > 0 \). The characteristic equation (1) has two distinct real roots, \( \lambda_1 \) and \( \lambda_2 \). The general solution of the system in question is expressed as

\[
x = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t},
\]

\[
y = C_1(\lambda_1 - a)e^{\lambda_1 t} + C_2(\lambda_2 - a)e^{\lambda_2 t},
\]

where \( C_1 \) and \( C_2 \) are arbitrary constants.

1.2. Let \( D < 0 \). The characteristic equation (1) has two complex conjugate roots, \( \lambda_{1,2} = \sigma \pm i\beta \). The general solution of the system in question is given by

\[
x = be^{\sigma t} \left[ C_1 \sin(\beta t) + C_2 \cos(\beta t) \right],
\]

\[
y = e^{\sigma t} \left\{ \left[ (\sigma - a)C_1 - \beta C_2 \right] \sin(\beta t) + \left[ \beta C_1 + (\sigma - a)C_2 \cos(\beta t) \right] \right\},
\]

where \( C_1 \) and \( C_2 \) are arbitrary constants.

1.3. Let \( D = 0 \) and \( a \neq d \). The characteristic equation (1) has two equal real roots, \( \lambda_1 = \lambda_2 \). The general solution of the system in question is written as

\[
x = 2b \left( C_1 + \frac{C_2}{a - d} + C_2 t \right) e^{\frac{(a + d) t}{2}},
\]

\[
y = \left[ (d - a)C_1 + C_2 + (d - a)C_2 t \right] e^{\frac{(a + d) t}{2}},
\]

where \( C_1 \) and \( C_2 \) are arbitrary constants.

1.4. Let \( a = d \neq 0 \) and \( b = 0 \). Solution:

\[
x = C_1 e^{at}, \quad y = (cC_1 t + C_2)e^{at}.
\]

1.5. Let \( a = d \neq 0 \) and \( c = 0 \). Solution:

\[
x = (bC_1 t + C_2)e^{at}, \quad y = C_1 e^{at}.
\]

2\(^o\). Case \( ad - bc = 0 \) and \( a^2 + b^2 > 0 \). The whole straight line \( ax + by = 0 \) consists of singular points. The original system of differential equations can be rewritten as

\[
x'_1 = ax + by, \quad y'_1 = k(ax + by).
\]
2.1. Let $a + bk \neq 0$. Solution:

$$x = bC_1 + C_2 e^{(a+bk)t}, \quad y = -aC_1 + kC_2 e^{(a+bk)t}.$$ 

2.2. Let $a + bk = 0$. Solution:

$$x = C_1(bkt - 1) + bC_2 t, \quad y = k^2 bC_1 t + (bk^2 t + 1)C_2.$$ 

Reference