1. $x'_t = ax + by, \quad y'_t = cx + dy.$

System of two linear homogeneous first-order constant-coefficient ordinary differential equations.

The characteristic equation is written as

$$\lambda^2 - (a+d)\lambda + ad - bc = 0 \tag{1}$$

and its discriminant is

$$D = (a-d)^2 + 4bc. \tag{2}$$

Consider several cases.

1°. Case $ad - bc \neq 0$. The origin of coordinates, $x = y = 0$, is the only stationary point; it is:

$$\begin{aligned} &\text{a node if } D = 0;\\ &\text{a node if } D > 0,\ \ ad - bc > 0;\\ &\text{a saddle if } D > 0,\ \ ad - bc < 0;\\ &\text{a focus if } D < 0,\ \ a + d \neq 0;\\ &\text{a center if } D < 0,\ \ a + d = 0. \end{aligned}$$

1.1. Let $D > 0$. The characteristic equation (1) has two distinct real roots, λ_1 and λ_2. The general solution of the system in question is expressed as

$$x = C_1 b e^{\lambda_1 t} + C_2 b e^{\lambda_2 t},$$
$$y = C_1(\lambda_1 - a)e^{\lambda_1 t} + C_2(\lambda_2 - a)e^{\lambda_2 t},$$

where C_1 and C_2 are arbitrary constants.

1.2. Let $D < 0$. The characteristic equation (1) has two complex conjugate roots, $\lambda_{1,2} = \sigma \pm i\beta$. The general solution of the system in question is given by

$$x = be^{\sigma t}\big[C_1 \sin(\beta t) + C_2 \cos(\beta t)\big],$$
$$y = e^{\sigma t}\big\{[(\sigma - a)C_1 - \beta C_2]\sin(\beta t) + [\beta C_1 + (\sigma - a)C_2 \cos(\beta t)\big],$$

where C_1 and C_2 are arbitrary constants.

1.3. Let $D = 0$ and $a \neq d$. The characteristic equation (1) has two equal real roots, $\lambda_1 = \lambda_2$. The general solution of the system in question is written as

$$x = 2b\left(C_1 + \frac{C_2}{a-d} + C_2 t\right)\exp\left(\frac{a+d}{2}t\right),$$
$$y = [(d-a)C_1 + C_2 + (d-a)C_2 t]\exp\left(\frac{a+d}{2}t\right),$$

where C_1 and C_2 are arbitrary constants.

1.4. Let $a = d \neq 0$ and $b = 0$. Solution:

$$x = C_1 e^{at}, \qquad y = (cC_1 t + C_2)e^{at}.$$

1.5. Let $a = d \neq 0$ and $c = 0$. Solution:

$$x = (bC_1 t + C_2)e^{at}, \qquad y = C_1 e^{at}.$$

2°. Case $ad - bc = 0$ and $a^2 + b^2 > 0$. The whole straight line $ax + by = 0$ consists of singular points. The original system of differential equations can be rewritten as

$$x'_t = ax + by, \qquad y'_t = k(ax + by).$$

2.1. Let $a + bk \neq 0$. Solution:

$$x = bC_1 + C_2 e^{(a+bk)t}, \qquad y = -aC_1 + kC_2 e^{(a+bk)t}.$$

2.2. Let $a + bk = 0$. Solution:

$$x = C_1(bkt - 1) + bC_2 t, \qquad y = k^2 bC_1 t + (bk^2 t + 1)C_2.$$

Reference

Kamke, E., *Differentialgleichungen: Lösungsmethoden und Lösungen, I, Gewöhnliche Differentialgleichungen*, B. G. Teubner, Leipzig, 1977.

