8. $x''_{tt} = ax + by, \quad y''_{tt} = cx + dy.$

System of two constant-coefficient second-order linear homogeneous differential equations.

The characteristic equation has the form

$$\lambda^4 - (a+d)\lambda^2 + ad - bc = 0.$$

1°. *Case* $ad - bc \neq 0$.

1.1. *Let* $(a-d)^2 + 4bc \neq 0$. *The characteristic equation has four distinct roots,* $\lambda_1, \dots, \lambda_4$. *The general solution of the system in question is expressed as*

$$x = C_1 b e^{\lambda_1 t} + C_2 b e^{\lambda_2 t} + C_3 b e^{\lambda_3 t} + C_4 b e^{\lambda_4 t},$$
$$y = C_1(\lambda_1^2 - a)e^{\lambda_1 t} + C_2(\lambda_2^2 - a)e^{\lambda_2 t} + C_3(\lambda_3^2 - a)e^{\lambda_3 t} + C_4(\lambda_4^2 - a)e^{\lambda_4 t},$$

where $C_1, \dots, C_4$ *are arbitrary constants.*

1.2. *Solution with* $(a-d)^2 + 4bc = 0$ *and* $a \neq d$:

$$x = 2C_1\left(bt + \frac{2bk}{a-d}\right)e^{kt/2} + 2C_2\left(bt - \frac{2bk}{a-d}\right)e^{-kt/2} + 2bC_3te^{kt/2} + 2bC_4te^{-kt/2},$$
$$y = C_1(d-a)te^{kt/2} + C_2(d-a)te^{-kt/2} + C_3[(d-a)t + 2k]e^{kt/2} + C_4[(d-a)t - 2k]e^{-kt/2},$$

where $C_1, \dots, C_4$ *are arbitrary constants, and* $k = \sqrt{2(a+d)}$.

1.3. *Solution with* $a = d \neq 0$ *and* $b = 0$:

$$x = 2\sqrt{a}\,C_1 e^{\sqrt{a}\,t} + 2\sqrt{a}\,C_2 e^{-\sqrt{a}\,t},$$
$$y = cC_1 te^{\sqrt{a}\,t} - cC_2 te^{-\sqrt{a}\,t} + C_3 e^{\sqrt{a}\,t} + C_4 e^{-\sqrt{a}\,t}.$$

1.4. *Solution with* $a = d \neq 0$ *and* $c = 0$:

$$x = bC_1 te^{\sqrt{a}\,t} - bC_2 te^{-\sqrt{a}\,t} + C_3 e^{\sqrt{a}\,t} + C_4 e^{-\sqrt{a}\,t},$$
$$y = 2\sqrt{a}\,C_1 e^{\sqrt{a}\,t} + 2\sqrt{a}\,C_2 e^{-\sqrt{a}\,t}.$$

2°. *Case* $ad - bc = 0$ *and* $a^2 + b^2 > 0$. *The original system can be rewritten in the form*

$$x''_{tt} = ax + by, \qquad y''_{tt} = k(ax + by).$$

2.1. *Solution with* $a + bk \neq 0$:

$$x = C_1 \exp\bigl(t\sqrt{a+bk}\,\bigr) + C_2 \exp\bigl(-t\sqrt{a+bk}\,\bigr) + C_3 bt + C_4 b,$$
$$y = C_1 k \exp\bigl(t\sqrt{a+bk}\,\bigr) + C_2 k \exp\bigl(-t\sqrt{a+bk}\,\bigr) - C_3 at - C_4 a.$$

2.2. *Solution with* $a + bk = 0$:

$$x = C_1 bt^3 + C_2 bt^2 + C_3 t + C_4,$$
$$y = kx + 6C_1 t + 2C_2.$$

Reference

Kamke, E., *Differentialgleichungen: Lösungsmethoden und Lösungen, I, Gewöhnliche Differentialgleichungen*, B. G. Teubner, Leipzig, 1977.